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NEW FIXED POINT THEOREMS ON BANACH GROUPS

A. R. POURMOSLEMI, [1] M. ADIB, [2] T. NAZARI *[3]

[1] *Department of Mathematics, Payame Noor University, Tehran, Iran.*
a_pourmoslemy@pnu.ac.ir
[2] *Department of Mathematics, Payame Noor University, Tehran, Iran.*
M.adib.pnu@gmail.com
[3] *Department of Mathematics, Payame Noor University, Tehran, Iran.*
taheree.nazari2222@gmail.com.

Abstract. In this paper, after introducing continuous, injective and sequentially convergent mapping on a group, a generalization of Kannan and Chatterjea's fixed point theorems on Banach groups is presented.

1. Introduction

Let $\mathcal{J}$ be a group and $\vartheta : \mathcal{J} \to \mathcal{J}$ be a mapping. An element $w \in \mathcal{J}$ is called a fixed point of ϑ if $\vartheta(w) = w$. Let $w_0 \in \mathcal{J}$ be an arbitrary element. Define the Picard iterative sequence $\{w_n\}$ in $\mathcal{J}$ as follows

$$w_{n+1} = \vartheta(w_n), \qquad (n = 0, 1, 2...).$$

We note that the convergence of this sequence plays a significant role in the existence of a fixed point for mapping ϑ. Define the n^{th} iterate of ϑ as $\vartheta^0 = I$ (the identity map) and $\vartheta^n = \vartheta^{n-1} o \vartheta$, for $n \geq 1$.

The fixed point theory is one of the most useful and essential tools of nonlinear analysis and its applications. The origin of fixed point

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* Speaker.

theory known as the Banach contraction principle. Banach contraction principle states that any contraction on a complete metric space has a unique fixed point [1]. In 1968 Kannan [6] proved that a self-mapping υ on a complete metric space $(\mathcal{T}, d)$ satisfying

$$d(t,s) \leq \eta(d(t,\upsilon(t)) + d(s,\upsilon(s)),$$

for all $t, s \in \mathcal{T}$ where $0 < \eta < \frac{1}{2}$ has a unique fixed point. A similar conclusion was also obtained by Chaterjee in 1972 [5]. Koparde and Wghmode [7] proved a fixed point theorem for a self-mapping υ on a complete metric space $(\mathcal{T}, d)$ satisfying the Kannan type condition

$$d^2(\upsilon(t), \upsilon(s)) \leq \eta(d^2(t,\upsilon(t)) + d^2(s,\upsilon(s)),$$

for all $t, s \in \mathcal{T}$ where $0 < \eta < \frac{1}{2}$.

Group-norms have played a role in topological groups [2]. The Birkhoff-Kakutanis metrization theorem for groups states that each first-countable Hausdorfgroup has a right invariant metric [3]. The term group-norm probably first appeared in Pettiss paper in 1950 [8]. In our further considerations, by applying sequentially convergent mappings, we will generalize this results in Banach groups.

Definition 1.1. [2] Let $\mathcal{J}$ be a group. A norm on a group $\mathcal{J}$ is a function $\|.\| : \mathcal{J} \to \mathbb{R}$ with the following properties:

(1) $\|w\| \geq 0, \ \textit{for all } w \in \mathcal{J};$
(2) $\|w\| = \|w^{-1}\|, \ \textit{for all } w \in \mathcal{J};$
(3) $\|wq\| \leq \|w\| + \|q\|, \ \textit{for all } w, q \in \mathcal{J};$
(4) $\|w\| = 0$ *implies that* $w = e$.

A normed group $(\mathcal{J}, \|.\|)$ is a group $\mathcal{J}$ equipped with a norm $\|.\|$. By setting $d(w,q) := \|w^{-1}q\|$, it is easy to see that norms are in bijection with left-invariant metrics on $\mathcal{J}$.

Definition 1.2. [2] A Banach group is a normed group $(\mathcal{J}, \|.\|)$, which is complete with respect to the metric

$$d(w,q) = \|wq^{-1}\|, \quad (w, q \in \mathcal{J}).$$

Definition 1.3. [4] Let $(\mathcal{T}, d)$ be a metric space. We call a mapping $\upsilon : \mathcal{T} \to \mathcal{T}$ is sequentially convergent if for each sequence $\{t_n\}$ that $\{\upsilon(t_n)\}$ is convergent then $\{t_n\}$ is also convergent.

2. Main results

Theorem 2.1. *Let* $(\mathcal{J}, \|.\|)$ *be a Banach group. Suppose that* $\vartheta : \mathcal{J} \to \mathcal{J}$ *be a map and* $\tau : \mathcal{J} \to \mathcal{J}$ *be a continuous, injection and sequentially* *convergent mapping. If $\eta > 0$, $\mu \geq 0$, $2\eta + \mu < 1$ and*

$$\|\tau\vartheta(w)(\tau\vartheta(q))^{-1}\| \leq \eta[\|\tau(w)(\tau\vartheta(w))^{-1}\| + \|\tau(q)(\tau\vartheta(q))^{-1}\|] \tag{2.1}$$
$$+ \mu\|\tau(w)\tau(q)^{-1}\| \tag{2.2}$$

for all $w, q \in \mathcal{J}$, then ϑ has a unique fixed point l. Moreover, for any $w_0 \in \mathcal{J}$ the sequence $\{\vartheta^n(w_0)\}$ converges to the l.

Proof. Suppose that $w_0 \in \mathcal{J}$ is given and the sequence $\{w_n\}$ be defined as the following $w_{n+1} = \vartheta(w_n)$ for $n = 0, 1, 2, \dots$. By taking $w = w_n$ and $q = w_{n-1}$ in (2.1) we get

$$\begin{aligned}\|\tau(w_{n+1})\tau(w_n)^{-1}\| &\leq \eta[\|\tau(w_{n+1})\tau(w_n)^{-1}\| + \|\tau(w_n)\tau(w_{n-1})^{-1}\|] \\ &\quad + \mu\|\tau(w_n)\tau(w_{n-1})^{-1}\|,\end{aligned}$$

then

$$\|\tau(w_{n+1})\tau(w_n)^{-1}\| \leq \alpha\|\tau(w_n)\tau(w_{n-1})^{-1}\|, \tag{2.3}$$

for each $n = 0, 1, 2, \dots$, and $0 < \alpha = \frac{\eta+\mu}{1-\eta} < 1$. By the inequality (2.3) we have

$$\|\tau(w_n)\tau(w_m)^{-1}\| \leq \frac{\alpha^m}{1-\alpha}\|\tau(w_1)\tau(w_0)^{-1}\|,$$

for all $n, m \in \mathbb{N}$ and $n > m$. Since $0 < \alpha < 1$ we conclude that the sequence $\{\tau(w_n)\}$ is Cauchy sequence. Completeness of $\mathcal{J}$ ensures that there exists $z \in \mathcal{J}$ such that $\lim_{n\to\infty} \tau(w_n) = z$. It implies that the sequence $\{w_n\}$ is also a convergent sequence, i.e. there exists $l \in \mathcal{J}$ such that $\lim_{n\to\infty} w_n = l$.

Since the mapping τ is continuous then $\lim_{n\to\infty} \tau(w_n) = \tau(l)$. Therefore, we have

$$\begin{aligned}\|\tau\vartheta(l)\tau(l)^{-1}\| &\leq \eta\|\tau(l)(\tau\vartheta(l))^{-1}\| + \eta\alpha^{n-1}\|\tau(w_1)\tau(w_0)^{-1}\| \\ &\quad + \mu\|\tau(l)\tau(w_{n-1})^{-1}\| + \alpha^n\|\tau(w_1)\tau(w_0)^{-1}\| \\ &\quad + \|\tau(w_{n+1})\tau(l)^{-1}\|.\end{aligned}$$

Letting $n \to \infty$ in the inequality above, since $0 < \alpha < 1$ and τ is continuous we conclude that $\|\tau\vartheta(l)\tau(l)^{-1}\| \leq \eta\|\tau\vartheta(l)\tau(l)^{-1}\|$. As $0 < \eta < 1$, then $\tau\vartheta(l) = \tau(l)$. But $\vartheta(l) = l$ since τ is an injection. For the uniqueness, we suppose that ϑ has two distinct fixed points $l, l_0 \in \mathcal{J}$. Then from (2.1) we have

$$\begin{aligned}\|\tau(l)\tau(l_0)^{-1}\| &= \|\tau\vartheta(l)\tau\vartheta(l_0)^{-1}\| \\ &\leq \eta[\|\tau(l)\tau\vartheta(l)\| + \|\tau(l_0)\tau\vartheta(l_0)^{-1}\|] + \mu\|\tau(l)\tau(l_0)^{-1}\| \\ &= \mu\|\tau(l)\tau(l_0)^{-1}\|.\end{aligned}$$

Since $0 < \mu < 1$, the last inequality implies that $\|\tau(l)\tau(l_0)^{-1}\| = 0$. As τ is an injection we have $l = l_0$. It follows immediately that $\lim_{n\to\infty} \vartheta^n(w_0) = l$ for any $w_0 \in \mathcal{J}$. □

Similarly, the following theorem could be solved

Theorem 2.2. *Let $(\mathcal{J}, \|.\|)$ be a Banach group, $\vartheta : \mathcal{J} \to \mathcal{J}$ be a map and $\tau : \mathcal{J} \to \mathcal{J}$ be continuous, injection and sequentially convergent mapping. If $\eta > 0$, $\mu \geq 0$, $2\eta + \mu < 1$ and*

$$\|\tau\vartheta(w)(\tau\vartheta(q))^{-1}\| \leq \eta[\|\tau(w)(\tau\vartheta(q))^{-1}\| + \|\tau(q)(\tau\vartheta(w))^{-1}\|] \tag{2.4}$$
$$+ \mu\|\vartheta(w)\tau(q)^{-1}\|, \tag{2.5}$$

for all $w, q \in \mathcal{J}$, then ϑ has a unique fixed point l. Moreover, for any $w_0 \in \mathcal{J}$ the sequence $\{\vartheta^n(w_0)\}$ converges to the l.

Theorem 2.3. *Let $(\mathcal{J}, \|.\|)$ be a Banach group, $\vartheta : \mathcal{J} \to \mathcal{J}$ and $\tau : \mathcal{J} \to \mathcal{J}$ be a mapping such that it is continuous, injection and sequentially convergent. If $\eta > 0$, $\mu \geq 0$, $2\eta + \mu < 1$ and*

$$\|\tau\vartheta(w)(\tau\vartheta(q))^{-1}\|^2 \leq \eta[\|\tau(w)(\tau\vartheta(w))^{-1}\|^2 + \|\tau(q)(\tau\vartheta(q))^{-1}\|^2] \tag{2.6}$$
$$+ \mu\|\tau(w)\tau(q)^{-1}\|^2 \tag{2.7}$$

for all $w, q \in \mathcal{J}$, then ϑ has a unique fixed point l. Moreover, for any $w_0 \in \mathcal{J}$ the sequence $\{\vartheta^n(w_0)\}$ converges to the l.

Proof. Suppose that $w_0 \in \mathcal{J}$ is given and the sequence $\{w_n\}$ be defined as the following $w_{n+1} = \vartheta(w_n)$ for $n = 0, 1, 2, \ldots$. By (2.6), we have

$$\|\tau(w_{n+1})\tau(w_n)^{-1}\|^2 \leq \eta[\|\tau(w_{n+1})\tau(w_n)^{-1}\|^2 + \|\tau(w_n)\tau(w_{n-1})^{-1}\|^2]$$
$$+ \mu\|\tau(w_n)\tau(w_{n-1})^{-1}\|^2.$$

Therefore,

$$\|\tau(w_{n+1})\tau(w_n)^{-1}\|^2 \leq \alpha\|\tau(w_n)\tau(w_{n-1})^{-1}\|^2, \tag{2.8}$$

for each $n = 0, 1, 2, \ldots$, and $0 < \alpha = (\frac{\eta+\mu}{1-\eta})^{\frac{1}{2}} < 1$. By the inequality (2.8) we have

$$\|\tau(w_n)\tau(w_m)^{-1}\|^2 \leq \frac{\alpha^m}{1-\alpha}\|\tau(w_1)\tau(w_0)^{-1}\|^2,$$

for all $n, m \in \mathbb{N}$ where $n > m$. Since $0 < \alpha < 1$ we conclude that the sequence $\{\tau(w_n)\}$ is a Cauchy sequence and there exists $z \in \mathcal{J}$ such that $\lim_{n\to\infty} \tau(w_n) = z$.

As $\tau : \mathcal{J} \to \mathcal{J}$ is sequentially convergent mapping and since the sequence $\{\tau(w_n)\}$ is convergent, it implies that the sequence $\{w_n\}$ is also convergent, i.e. there exists $l \in \mathcal{J}$ so that $\lim_{n\to\infty} w_n = l$.

Since the mapping τ is continuous we see that $\lim_{n\to\infty} \tau(w_n) = \tau(l)$. Now, we show that l is the unique fixed point of ρ. By (2.6), we have

$$\begin{aligned}\|\tau\vartheta(l)\tau(l)^{-1}\| &\leq \|\tau(l)\tau(w_{n+1})^{-1}\| + \|\tau(w_{n+1})(\tau\vartheta(l))^{-1}\|\\ &= \|\tau(l)\tau(w_{n+1})^{-1}\| + \|\tau\vartheta(w_n)(\tau\vartheta(l))^{-1}\|\\ &\leq \|\tau(l)\tau(w_{n+1})^{-1}\| + [\eta[\|\tau(w_n)(\tau\vartheta(w_n))^{-1}\|^2\\ &+ \|\tau(l)(\tau\vartheta(l))^{-1}\|^2] + \mu\|\tau(w_n)\tau(l)^{-1}\|^2]^{\frac{1}{2}}\\ &= \|\tau(l)\tau(w_{n+1})^{-1}\| + [\eta[\|\tau(w_n)(\tau(w_{n+1}))^{-1}\|^2\\ &+ \|\tau(l)(\tau\vartheta(l))^{-1}\|^2] + \mu\|\tau(w_n)\tau(l)^{-1}\|^2]^{\frac{1}{2}},\end{aligned}$$

for each $n \in \mathbb{N}$. For $n \to \infty$, the latter is transformed as the following $\|\tau\vartheta(l)\tau(l)^{-1}\| \leq \eta^{\frac{1}{2}}\|\tau\vartheta(l)\tau(l)^{-1}\|$. But, $\eta < 1$. Therefore, $\|\tau\vartheta(l)\tau(l)^{-1}\| = 0$. To see the uniqueness of the fixed point of ϑ, let $l, l_0 \in \mathcal{J}$ be two fixed points on ϑ. Using (2.6), we have

$$\begin{aligned}\|\tau(l)\tau(l_0)^{-1}\|^2 &= \|\tau\vartheta(l)(\tau\vartheta(l_0))^{-1}\|^2\\ &\leq \eta[\|\tau(l)(\tau\vartheta(l))^{-1}\|^2 + \|\tau(l_0)(\tau\vartheta(l_0))^{-1}\|^2]\\ &+ \mu\|\tau(l)\tau(l_0)^{-1}\|^2\\ &= \eta[\|\tau(l)\tau(l)^{-1}\|^2 + \|\tau(l_0)\tau(l_0)^{-1}\|^2] + \mu\|\tau(l)\tau(l_0)^{-1}\|^2\\ &= \mu\|\tau(l)\tau(l_0)^{-1}\|^2,\end{aligned}$$

and since $0 < \mu < 1$ the latter inequality implies that $\|\tau(l)\tau(l_0)^{-1}\| = 0$. But, τ is an injection, and thus $l = l_0$. Finally, for each $w_0 \in \mathcal{J}$ the sequence $\{\vartheta^n(w_0)\}$ converges to the unique fixed point on ϑ. □

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