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FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS IN A COMPLETE METRIC SPACE

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Abstract. In this paper, two fixed point theorems for generalized contractions with constants in a complete metric space is presented.

1. Introduction

One of the most popular tools for solving problems in nonlinear analysis and its applications is the Banach fixed point theorem [1]. In 1968 Kannan [7] and in 1972 Chatterjea [4] studied contractive mappings which give a unique fixed point on a complete metric space. In this paper we use Picard iteration given as following:
Let $(\mathcal{T}, \mathfrak{d})$ be a metric space and $\theta : \mathcal{T} \to \mathcal{T}$ be a mapping. For any $t_0 \in \mathcal{T}$, the sequence $\{t_n\} \subset \mathcal{T}$ given by

$$t_n = \theta(t_{n-1}) = \theta^n(t_0) \quad n = 0, 1, 2, ...$$

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is called the sequence of successive approximations with the initial value t_0 [2].

Let $(\mathcal{T}, \mathfrak{d})$ be a metric space. A mapping $\theta : \mathcal{T} \to \mathcal{T}$, is said to be a contraction if there exists $\eta \in [0, 1)$ such that for all $t, s \in \mathcal{T}$

$$\mathfrak{d}(\theta(t), \theta(s)) \leq \eta \mathfrak{d}(t, s). \tag{1.1}$$

If the metric space $(\mathcal{T}, \mathfrak{d})$ is complete then the mapping satisfying (1.1) has a unique fixed point [1]. Also, Kannan [7] established the following result:
If a mapping $\theta : \mathcal{T} \to \mathcal{T}$ where $(\mathcal{T}, \mathfrak{d})$ is a complete metric space, satisfies the inequality

$$\mathfrak{d}(\theta(t), \theta(s)) \leq \eta[\mathfrak{d}(t, \theta(t)) + \mathfrak{d}(s, \theta(s))], \tag{1.2}$$

where $\eta \in [0, \frac{1}{2})$ and $t, s \in \mathcal{T}$. Then, θ has a unique fixed point. The mappings satisfying (1.2) are called Kannan type mappings. The significance of Kannan's theorem appeared in Subrahmanyam paper [8]. He showed that a metric space is complete if and only if every Kannan mapping has a fixed point. Banach contractions do not have this property; Connell in [5] has given an example of metric space $\mathcal{T}$ that is not complete but every Banach contraction on $\mathcal{T}$ has a fixed point.
In 2011 Moradi and Alimohammadi [6] introduced new extensions of Kannan fixed point theorem on generalized metric spaces as following:

Theorem 1.1. *Let $(\mathcal{T}, \mathfrak{d})$ be a complete generalized metric space and $\theta, \vartheta : \mathcal{T} \to \mathcal{T}$ be mappings such that θ is continuous, one-to-one and sequentially convergent. If $0 \leq \eta < \frac{1}{2}$ and*

$$\mathfrak{d}(\theta\vartheta(t)), \theta\vartheta(s) \leq \eta[\mathfrak{d}(\theta(t), \theta\vartheta(t)) + \mathfrak{d}(\theta(s), \theta\vartheta(s))],$$

for all $t, s \in \mathcal{T}$, then ϑ has a unique fixed point.

2. Main results

In this section, by applying sequentially convergent mappings, we generalize contractions with constants in a complete metric space and prove two main theorems which are generalization of very recent results.

Definition 2.1. [3] Let $(\mathcal{T}, \mathfrak{d})$ be a metric space. We call a mapping $\upsilon : \mathcal{T} \to \mathcal{T}$ is sequentially convergent if for each sequence $\{t_n\}$ that $\{\upsilon(t_n)\}$ is convergent then $\{t_n\}$ is also convergent.

Let Ψ be the class of all nondecreasing continuous functions $\sigma : [0, +\infty) \to [0, +\infty)$ such that $\sigma(t) = 0$ if and only if $t = 0$.

Definition 2.2. A function $\sigma : [0, \infty) \to [0, \infty)$ is called a subadditive altering distance function if

(1) $\sigma \in \Psi$,
(2) $\sigma(a+b) \leq \sigma(a) + \sigma(b)$ for all $a, b \in [0, \infty)$.

Example 2.3. It is easy to see that the functions $\sigma_1(a) = a^{\frac{1}{n}}$, $n \in \mathbb{N}$, and $\sigma_2(a) = \log(a+1)$, $a \geq 0$ are such subadditive altering distance functions.

In the following theorem we prove a new extension of Kannan's theorem.

Theorem 2.4. *Let $(\mathcal{T}, \mathfrak{d})$ be a complete metric space and $\theta, \vartheta : \mathcal{T} \to \mathcal{T}$ be mappings such that θ is continuous, injection and sequentially convergent. Consider σ be a subadditive altering distance function. If $\eta > 0$, $\mu \geq 0$, $2\eta + \mu < 1$, and*

$$\begin{aligned}\sigma(\mathfrak{d}(\theta\vartheta(t), \theta\vartheta(s)) &\leq \eta[\sigma(\mathfrak{d}(\theta(t), \theta\vartheta(t))) + \sigma(\mathfrak{d}(\theta(s), \theta\vartheta(s)))] \\ &\quad + \mu\sigma(\mathfrak{d}(\theta(t), \theta(s))), \qquad (2.1)\end{aligned}$$

for all $t, s \in \mathcal{T}$, then ϑ has a unique fixed point.

Proof. Since $\sigma^{-1}(0) = \{0\}$, for every $\varepsilon > 0$, $\sigma(\varepsilon) > 0$. Suppose that $t_0 \in \mathcal{T}$ is given and the sequence $\{t_n\}$ be defined as $t_{n+1} = \vartheta(t_n)$ for $n = 0, 1, 2, \ldots$. By taking $t = t_{n-1}$ and $s = t_n$ in (2.1) we get

$$\begin{aligned}\sigma(\mathfrak{d}(\theta(t_n), \theta(t_{n+1}))) &= \sigma(\mathfrak{d}(\theta\vartheta(t_{n-1}), \theta\vartheta(t_n)) \\ &\leq \eta[\sigma(\mathfrak{d}(\theta(t_{n-1}), \theta\vartheta(t_{n-1}))) + \sigma(\mathfrak{d}(\theta(t_n), \theta\vartheta(t_n)))] \\ &\quad + \mu\sigma(\mathfrak{d}(\theta(t_{n-1}), \theta(t_n))).\end{aligned}$$

Therefore,

$$\sigma(\mathfrak{d}(\theta(t_n), \theta(t_{n+1}))) \leq \alpha\sigma(\mathfrak{d}(\theta(t_{n-1}), \theta(t_n))) \qquad (2.2)$$

for each $n = 0, 1, 2, \ldots$, and $0 < \alpha = \frac{\eta+\mu}{1-\eta} < 1$.
Then

$$\begin{aligned}\sigma(\mathfrak{d}(\theta(t_n), \theta(t_{n+1}))) &\leq \alpha\sigma(\mathfrak{d}(\theta(t_{n-1}), \theta(t_n))) \\ &\leq \alpha^2\sigma(\mathfrak{d}(\theta(t_{n-2}), \theta(t_{n-1}))) \\ &\leq \ldots \leq \alpha^n\sigma(\mathfrak{d}(\theta(t_0), \theta(t_1))).\end{aligned}$$

By (2.2), for all $m, n \in \mathbb{N}$ that $n < m$, we have

$$\begin{aligned}\sigma(\mathfrak{d}(\theta(t_m), \theta(t_n))) &\leq \sigma(\mathfrak{d}(\theta(t_m), \theta(t_{m-1})) + \mathfrak{d}(\theta(t_{m-1}), \theta(t_{m-2})) \\ &\quad + \ldots + \mathfrak{d}(\theta(t_{n+1}), \theta(t_n))) \\ &\leq \sigma(\mathfrak{d}(\theta(t_m), \theta(t_{m-1}))) + \sigma(\mathfrak{d}(\theta(t_{m-1}), \theta(t_{m-2}))) \\ &\quad + \ldots + \sigma(\mathfrak{d}(\theta(t_{n+1}), \theta(t_n))) \\ &\leq (\alpha^{m-1} + \alpha^{m-2} + \ldots + \alpha^n)\sigma(\mathfrak{d}(\theta(t_0), \theta(t_1))).\end{aligned}$$

So,

$$\sigma(\mathfrak{d}(\theta(t_m), \theta(t_n))) \leq \frac{\alpha^n}{1-\alpha}\sigma(\mathfrak{d}(\theta(t_0), \theta(t_1))).$$

Then $\{\theta(t_n)\}$ is a Cauchy sequence. Since $\mathcal{T}$ is a complete metric space, there exists $z \in \mathcal{T}$ such that

$$\lim_{n\to\infty} \theta(t_n) = z.$$

Further, the mapping $\theta : \mathcal{T} \to \mathcal{T}$ is sequentially convergent. Since the sequence $\theta(t_n)$ is convergent, it implies that the sequence t_n is also convergent, i.e. there exists $w \in \mathcal{T}$ such that $\lim_{n\to\infty} t_n = w$. Since θ is continuous, $\lim_{n\to\infty} \theta(t_n) = \theta(w)$. Thus,

$$\begin{aligned}
\sigma(\mathfrak{d}(\theta\vartheta(w), \theta(t_{n+1}))) &= \sigma(\mathfrak{d}(\theta\vartheta(w), \theta\vartheta(t_n))) \\
&\leq \eta[\sigma(\mathfrak{d}(\theta(w), \theta\vartheta(w))) + \sigma(\mathfrak{d}(\theta(t_n), \theta\vartheta(t_n)))] \\
&+ \mu\sigma(\mathfrak{d}(\theta(w), \theta(t_n))) \\
&= \eta[\sigma(\mathfrak{d}(\theta(w), \theta\vartheta(w))) + \sigma(\mathfrak{d}(\theta(t_n), \theta(t_{n+1})))] \\
&+ \mu\sigma(\mathfrak{d}(\theta(w), \theta(t_n))). \qquad (2.3)
\end{aligned}$$

Now, letting $n \to \infty$ in (2.3) we get

$$\sigma(\mathfrak{d}(\theta\vartheta(w), \theta(w))) \leq \eta[\sigma(\mathfrak{d}(\theta(w), \theta\vartheta(w))) + \sigma(0)] + \mu\sigma(0).$$

But $\mathfrak{d}(\theta\vartheta(w), \theta(w)) = 0$, since $\sigma^{-1}(0) = 0$ and $0 < \eta < 1$. θ is injection, and thus $\vartheta(w) = w$. The latter actually means that the mapping ϑ has a fixed point.
To prove uniqueness, let v be another fixed point of ϑ. Then by (2.2), we have

$$\begin{aligned}
\sigma(\mathfrak{d}(\theta(w), \theta(v))) &= \sigma(\mathfrak{d}(\theta\vartheta(w), \theta\vartheta(v))) \\
&\leq \eta[\sigma(\mathfrak{d}(\theta(w), \theta\vartheta(w))) + \sigma(\mathfrak{d}(\theta(v), \theta\vartheta(v)))] \\
&+ \mu\sigma(\mathfrak{d}(\theta(w), \theta(v))) \\
&= \eta[\sigma(\mathfrak{d}(\theta(w), \theta(w))) + \sigma(\mathfrak{d}(\theta(v), \theta(v)))] \\
&+ \mu\sigma(\mathfrak{d}(\theta(w), \theta(v))) \\
&= \mu\sigma(\mathfrak{d}(\theta(w), \theta(v))).
\end{aligned}$$

Since $0 < \mu < 1$, the last inequality implies that $\sigma(\mathfrak{d}(\theta(w), \theta(v))) = 0$, i.e. $\theta(w) = \theta(v)$. Finally, the injectivity of θ implies $w = v$. $\square$

By taking $\mu = 0$ in Theorem 2.4, we get the following theorem.

Theorem 2.5. *Let $(\mathcal{T}, \mathfrak{d})$ be a complete metric space and $\theta, \vartheta : \mathcal{T} \to \mathcal{T}$ be mappings such that θ is continuous, injection and sequentially* *convergent. If* $0 \leq \eta < \frac{1}{2}$, $\sigma \in \Psi$ *and*

$$\sigma(\mathfrak{d}(\theta\vartheta(t), \theta\vartheta(s)) \leq \eta[\sigma(\mathfrak{d}(\theta(t), \theta\vartheta(t))) + \sigma(\mathfrak{d}(\theta(s), \theta\vartheta(s)))],$$

for all $t, s \in \mathcal{T}$, *then* ϑ *has a unique fixed point.*

Similarly, the following theorem could be solved.

Theorem 2.6. *Let* $(\mathcal{T}, \mathfrak{d})$ *be a complete metric space and* $\theta, \vartheta : \mathcal{T} \to \mathcal{T}$ *be mappings such that* θ *is continuous, injection and sequentially convergent. If* $\eta > 0$, $\mu \geq 0$, $2\eta + \mu < 1$, *and*

$$\sigma(\mathfrak{d}(\theta\vartheta(t), \theta\vartheta(s)) \leq \eta[\sigma(\mathfrak{d}(\theta(t), \theta\vartheta(s))) + \sigma(\mathfrak{d}(\theta(s), \theta\vartheta(t)))] + \mu\sigma(\mathfrak{d}(\theta(t), \theta(s))),$$

for all $t, s \in \mathcal{T}$, *then* ϑ *has a unique fixed point.*

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