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WEIGHTED GENERALIZED SHIFT ISOMORPHISMS ON ℓ^p

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Abstract. In the following text we classify all "appropriate" weighted generalized shift isomorphisms on $\ell^p(\Gamma)$ for nonempty set Γ and $p \in [1, \infty]$.

1. Introduction

Let's recall that for nonempty set Γ and $p \in [1, \infty)$ we have Banach spaces $\ell^p(\Gamma) = \{(x_\alpha)_{\alpha\in\Gamma} \in \mathbb{C}^\Gamma : \sum\limits_{\alpha\in\Gamma} |x_\alpha|^p < \infty\}$ equipped with norm $||(x_\alpha)_{\alpha\in\Gamma}||_p = \left(\sum\limits_{\alpha\in\Gamma} |x_\alpha|^p\right)^{\frac{1}{p}}$ (for $(x_\alpha)_{\alpha\in\Gamma} \in \ell^p(\Gamma)$) and $\ell^\infty(\Gamma) = \{(x_\alpha)_{\alpha\in\Gamma} \in \mathbb{C}^\Gamma : \sup\limits_{\alpha\in\Gamma} |x_\alpha| < \infty\}$ equipped with norm $||(x_\alpha)_{\alpha\in\Gamma}||_\infty = \sup\limits_{\alpha\in\Gamma} |x_\alpha|$ (for $(x_\alpha)_{\alpha\in\Gamma} \in \ell^\infty(\Gamma)$). Moreover as it has been mentioned in [2], for $w = (w_\alpha)_{\alpha\in\Gamma} \in \mathbb{C}$ and $\varphi : \Gamma \to \Gamma$ one may consider "weighted generalized shift" $\sigma_{\varphi,w} : \mathbb{C}^\Gamma \to \mathbb{C}^\Gamma$ with $\sigma_{\varphi,w}((x_\alpha)_{\alpha\in\Gamma}) = (w_\alpha x_{\varphi(\alpha)})_{\alpha\in\Gamma}$. We also know the following statements for $t \in [1, \infty]$ are equivalent [2, Theorem 2.1]:

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- $\sigma_{\varphi,w}(\ell^t(\Gamma)) \subseteq \ell^t(\Gamma)$,
- $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is continuous and $||\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}|| = \sup\{||(w_\alpha)_{\alpha\in\varphi^{-1}(\beta)}||_t : \beta \in \varphi(\Gamma)\} < +\infty$.

Weighted generalized shifts are generalization of generalized shifts [1] and weighted shifts.
In the following text for appropriate w we classify all isometric isomorphism weighted generalized shifts $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$.

2. Isomorphism weighted generalized shifts

Consider $t \in [1,\infty]$, $\varphi : \Gamma \to \Gamma$ and $w = (w_\alpha)_{\alpha\in\Gamma}$ such that $\sup\{||(w_\alpha)_{\alpha\in\varphi^{-1}(\beta)}||_t : \beta \in \varphi(\Gamma)\} < +\infty$.

Theorem 2.1. *$\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is bijective if and only if*

- *$\varphi : \Gamma \to \Gamma$ is bijective,*
- *for all $\alpha \in \Gamma$ we have $w_\alpha \neq 0$,*
- *$\sup\limits_{\alpha\in\Gamma} |w_\alpha| < +\infty$ and $\sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|} < +\infty$.*

Proof. "⇒" Suppose $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is bijective. We have the following steps:
Step 1. for all $\alpha \in \Gamma$ we have $w_\alpha \neq 0$. For $\beta \in \Gamma$ if $w_\beta = 0$, then $\sigma_{\varphi,w}(\ell^t(\Gamma)) = \sigma_{\varphi,w}(\ell^t(\Gamma)) \cap \ell^t(\Gamma) = \{(w_\alpha x_{\varphi(\alpha)})_{\alpha\in\Gamma} : (x_\alpha)_{\alpha\in\Gamma} \in \ell^t(\Gamma)\} \cap \ell^t(\Gamma) \subseteq \{(y_\alpha)_{\alpha\in\Gamma} \in \ell^t(\Gamma) : y_\beta = 0\} \subsetneqq \ell^t(\Gamma)$ and $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is not surjective which is a contradiction (since we have supposed $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is bijective). Thus:

$$\forall \alpha \in \Gamma \;\; w_\alpha \neq 0 \,.$$

Step 2. $\varphi : \Gamma \to \Gamma$ is injective. If for distinct $\beta, \theta \in \Gamma$ we have $\varphi(\beta) = \varphi(\theta)$, then $\sigma_{\varphi,w}(\ell^t(\Gamma)) \subseteq \{(y_\alpha)_{\alpha\in\Gamma} \in \ell^t(\Gamma) : r_\theta y_\beta = r_\beta y_\theta\} \subsetneqq \ell^t(\Gamma)$ and $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is not surjective which is in contradiction with our hypothesis, thus $\varphi : \Gamma \to \Gamma$ is one–to–one.
Step 3. $\varphi : \Gamma \to \Gamma$ is surjective. If $\beta \in \Gamma \setminus \varphi(\Gamma)$, then for $\delta^\beta_\alpha = 0$ if $\alpha \neq \beta$ and $\delta^\beta_\beta = 1$ we have $(\delta^\beta_\alpha)_{\alpha\in\Gamma} \in \ell^t(\Gamma)$ with $\sigma_{\varphi,w}((\delta^\beta_\alpha)_{\alpha\in\Gamma}) = (0)_{\alpha\in\Gamma} = \sigma_{\varphi,w}((0)_{\alpha\in\Gamma})$ which is in contradiction with injevtivity of $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$. hence $\varphi : \Gamma \to \Gamma$ is surjective.
Step 4. $\sup\limits_{\alpha\in\Gamma} |w_\alpha| < +\infty$. Since $\sigma_{\varphi,w}(\ell^t(\Gamma)) \subseteq (\ell^t(\Gamma))$ by [2, Theorem 2.1] we have $\sup\{||(w_\alpha)_{\alpha\in\varphi^{-1}(\beta)}||_t : \beta \in \varphi(\Gamma)\} < +\infty$. On the other

hand

$$\begin{aligned}\sup_{\beta\in\varphi(\Gamma)} ||(w_\alpha)_{\alpha\in\varphi^{-1}(\beta)}||_t &= \sup_{\beta\in\Gamma} ||(w_\alpha)_{\alpha\in\varphi^{-1}(\varphi(\beta))}||_t \\ &\underset{\text{steps 2,3}}{=} \sup_{\beta\in\Gamma} ||(w_\alpha)_{\alpha\in\{\beta\}}||_t = \sup_{\beta\in\Gamma} |w_\beta|\end{aligned}$$

hence $\sup\limits_{\beta\in(\Gamma)} |w_\beta| < +\infty$.

Step 5. $\sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|} < +\infty$. For $\alpha\in\Gamma$ let $u_\alpha := \frac{1}{w_{\varphi^{-1}(\alpha)}}$ and $u := (u_\alpha)_{\alpha\in\Gamma}$. Consider $x = (x_\alpha)_{\alpha\in\Gamma} \in \ell^t(\Gamma)$ since $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is bijective, there exists $z = (z_\alpha)_{\alpha\in\Gamma} \in \ell^t(\Gamma)$ with $\sigma_{\varphi,w}((z_\alpha)_{\alpha\in\Gamma}) = (x_\alpha)_{\alpha\in\Gamma}$ hence for all $\alpha\in\Gamma$ we have $w_\alpha z_{\varphi(\alpha)} = x_\alpha$. So for all $\alpha\in\Gamma$ we have $z_\alpha = \frac{1}{w_{\varphi^{-1}(\alpha)}} x_{\varphi^{-1}(\alpha)} = u_\alpha x_{\varphi^{-1}(\alpha)}$, which leads to $\sigma_{\varphi^{-1},u}(x) = z \in \ell^t(\Gamma)$. Using $\sigma_{\varphi^{-1},u}(\ell^t(\Gamma)) \subseteq \ell^t(\Gamma)$ we have $\sup\{||(u_\alpha)_{\alpha\in(\varphi^{-1})^{-1}(\beta)}||_t : \beta\in\varphi^{-1}(\Gamma)\} < +\infty$. Using the same method as in Step 4, we have $\sup\{||(u_\alpha)_{\alpha\in(\varphi^{-1})^{-1}(\beta)}||_t : \beta\in\varphi^{-1}(\Gamma)\} = \sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|}$ which completes the proof of Step 5.

"$\Leftarrow$" Now suppose $\varphi:\Gamma\to\Gamma$ is bijective, for all $\alpha\in\Gamma$ we have $w_\alpha \neq 0$ and both $\sup\limits_{\alpha\in\Gamma} |w_\alpha|$, $\sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|}$ are finite. Let $u_\alpha = \frac{1}{w_{\varphi^{-1}(\alpha)}}$ and $u := (u_\alpha)_{\alpha\in\Gamma}$. Using $\sup\{||(w_\alpha)_{\alpha\in\varphi^{-1}(\beta)}||_t : \beta\in\varphi(\Gamma)\} = \sup\limits_{\alpha\in\Gamma} |w_\alpha| < +\infty$ and $\sup\{||(u_\alpha)_{\alpha\in(\varphi^{-1})^{-1}(\beta)}||_t : \beta\in\varphi^{-1}(\Gamma)\} = \sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|} < \infty$, we have:

$$\sigma_{\varphi,w}(\ell^t(\Gamma)) \subseteq \ell^t(\Gamma), \sigma_{\varphi^{-1},u}(\ell^t(\Gamma)) \subseteq \ell^t(\Gamma)$$

It is easy to verify $\left(\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}\right)^{-1} = \sigma_{\varphi^{-1},u} \restriction_{\ell^t(\Gamma)}$ which leads to the desired result. $\square$

By Theorem 2.1 and[2, Theorem2.1] we have the following corollary:

Corollary 2.2. *the following statements are equivalent:*
1. $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ *is bijective,*
2. $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ *is an isomorphism,*
3. The following conditions hold:

- $\varphi:\Gamma\to\Gamma$ *is bijective,*
- *for all* $\alpha\in\Gamma$ *we have* $w_\alpha \neq 0$*,*
- $\sup\limits_{\alpha\in\Gamma} |w_\alpha| < +\infty$ *and* $\sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|} < +\infty$*.*

3. Isometric isomorphism weighted generalized shifts

Now we are ready to classify all isometric isomorphism weighted generalized shifts on $\ell^t(\Gamma)$.

Theorem 3.1. *Consider $t \in [1,\infty]$, $\Gamma \neq \varnothing$, $\varphi : \Gamma \to \Gamma$ and $w = (w_\alpha)_{\alpha\in\Gamma} \in \mathbb{C}^\Gamma$. Then $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is an isometry isomorphism if and only if*

- *$\varphi : \Gamma \to \Gamma$ is bijective,*
- *for all $\alpha \in \Gamma$ we have $|w_\alpha| = 1$.*

Proof. First suppose $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is an isometry isomorphism, then $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is bijective and by Theorem 2.1 $\varphi : \Gamma \to \Gamma$ is bijective, moreover for all $\alpha \in \Gamma$ we have $w_\alpha \neq 0$ also $\sup\limits_{\alpha\in\Gamma} |w_\alpha| < +\infty$ and $\sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|} < +\infty$. By [2, Theorem 2.1] we have $||\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)} || = \sup\{||(w_\alpha)_{\alpha\in\varphi^{-1}(\beta)}||_t : \beta \in \varphi(\Gamma)\}$, by the same argument as in the proof of Step 4 in Theorem 2.1 we have $\sup\{||(w_\alpha)_{\alpha\in\varphi^{-1}(\beta)}||_t : \beta \in \varphi(\Gamma)\} = \sup\limits_{\alpha\in\Gamma} |w_\alpha|$. Since $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is isometry we have $1 = ||\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)} || = \sup\limits_{\alpha\in\Gamma} |w_\alpha|$. Using the same argument as in the proof of Theorem 2.1 we have $\left(\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}\right)^{-1} = \sigma_{\varphi^{-1},u} \restriction_{\ell^t(\Gamma)}$ for $u = (\frac{1}{w_{\varphi^{-1}(\alpha)}})_{\alpha\in\Gamma}$ however $\sigma_{\varphi^{-1},u} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is isometry too, hence $1 = ||\sigma_{\varphi^{-1},u} \restriction_{\ell^t(\Gamma)} || = \sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|}$. By $\sup\limits_{\alpha\in\Gamma} |w_\alpha| = 1 = \sup\limits_{\alpha\in\Gamma} \frac{1}{|w_\alpha|}$ we have $|w_\alpha| = 1$ for all $\alpha \in \Gamma$.
Now suppose $\varphi : \Gamma \to \Gamma$ is bijective and for all $\alpha \in \Gamma$ we have $|w_\alpha| = 1$. By Theorem 2.1, $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is bijective. Consider $x = (x_\alpha)_{\alpha\in\Gamma} \in \ell^t(\Gamma)$ we have:

$$\begin{aligned} ||\sigma_{\varphi,w}(x)||_t &= ||(w_\alpha x_{\varphi(\alpha)})_{\alpha\in\Gamma}||_t = ||(\,|w_\alpha x_{\varphi(\alpha)}|\,)_{\alpha\in\Gamma}||_t \\ &\overset{(\forall\alpha\in\Gamma\ |w_\alpha|=1)}{=} ||(\,|x_{\varphi(\alpha)}|\,)_{\alpha\in\Gamma}||_t \\ &\overset{\varphi:\Gamma\to\Gamma \text{ is bijective}}{=} ||(\,|x_\alpha|\,)_{\alpha\in\Gamma}||_t = ||x||_t \end{aligned}$$

and $\sigma_{\varphi,w} \restriction_{\ell^t(\Gamma)}: \ell^t(\Gamma) \to \ell^t(\Gamma)$ is an isometry. □

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