

The Extended Abstracts of  
The 6<sup>th</sup> Seminar on Functional Analysis and its Applications  
4-5th March 2020, University of Isfahan, Iran

## HOW THE CONCEPT OF HYPERCYCLICITY WAS LOCALIZED?

MEYSAM ASADIPOUR \*

*Department of Mathematics, College of Sciences, Yasouj University,  
Yasouj, 75918-74934, Iran  
Asadipour@yu.ac.ir*

ABSTRACT. G. Costakis and A. Manoussos localized the concept of hypercyclicity by introducing certain sets, which they called  $\mathcal{J}$ -sets. In this paper we give a description of  $\mathcal{J}$ -sets through the use of open sets with an additional assumption.

### 1. INTRODUCTION

Let  $X$  be a Banach space over the field  $\mathbb{C}$  of complex numbers. In what follows, the symbol  $\mathbb{N}$  denotes the set of all positive integers,  $M$  is a subspace of the underlying Banach space  $X$  and symbol  $T$  stands for a bounded linear operator acting on  $X$ . Consider any vector  $x \in X$ , the symbol  $Orb(T, x)$  denotes the orbit of  $x$  under operator  $T$ , i. e.  $Orb(T, x) = \{T^n x : n \in \mathbb{N} \cup \{0\}\}$ . If  $Orb(T, x)$  is dense in  $X$ , then the operator  $T$  is called hypercyclic and the vector  $x$  is a hypercyclic vector for  $T$ . Observe that in this case, the underlying Banach space  $X$  should be separable. Then it is well known and easy to show that an operator  $T$  is hypercyclic if and only if  $T$  is topologically transitive, to be precise, for every pair of nonempty open subsets  $U, V$  of  $X$ , there exists a non-negative integer  $n$  such that  $T^n(U) \cap V \neq \emptyset$ . See the articles

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2010 *Mathematics Subject Classification.* Primary 47A16; Secondary 37B99, 54H20.

*Key words and phrases.* Hypercyclicity, Subspace hypercyclicity,  $\mathcal{J}$ -sets.

\* Speaker.

[?] and [?] with two books [?], [?] for some details on hypercyclicity and supercyclicity and related properties.

G.Costakis and A.Manoussos in [?] somehow localized the concept of hypercyclicity by introducing certain sets ,which is called  $\mathcal{J}$ -sets. To be precise for a given vector  $x \in X$  and an operator  $T$ , they defined

$$\mathcal{J}(x) = \{y \in X : \text{there exists a strictly increasing sequence of positive integers } \{k_n\} \text{ and a sequence } \{x_n\} \subset X \text{ such that } x_n \longrightarrow x \text{ and } T^{k_n}x_n \longrightarrow y\}.$$

It is worthwhile to mention that in [?], several useful results has been proven in relation to this concept.

Madore and Martinez-Avendano [?] proposed the notion of subspace hypercyclicity. Operator  $T$  for subspace  $M$  of  $X$  is subspace-hypercyclic or  $M$ -hypercyclic if there is a vector  $x \in X$ , so that  $\overline{Orb(T, x) \cap M} = M$ . In the above study, the subspace-transitivity was introduced and it was confirmed that  $M$ -hypercyclicity is derived from  $M$ -transitivity. Le [?] showed that  $M$ -transitivity is not equivalent to  $M$ - hypercyclicity. One can seen more information about concept of the subspace hypercyclicity in [?].

In the following, we will elucidate who we have a description of  $\mathcal{J}(x)$  through the use of open sets with an additional assumption. Also, we propose to combine both the concepts of  $\mathcal{J}$ -class and subspace transitivity and we'll ask a specific question.

## 2. BACKGROUND AND MAIN RESULT

**Definition 2.1.** An operator  $T$  is called subspace topologically transitive with respect to the subspace  $M$ , if for any pair  $U, V$  of nonempty relatively open subsets of  $M$ , there exists some integers  $n \geq 0$ , such that  $T^n(U) \cap V \neq \emptyset$  and  $T^n(M) \subseteq M$ .

Note that If  $M = X$ , then the famous definition of topologically transitivity is obtained.

**Definition 2.2.** Let  $T$  be an operator. For every  $x \in X$  the set

$$\mathcal{J}(x) = \{y \in X : \text{there exists a strictly increasing sequence of positive integers } \{k_n\} \text{ and a sequence } \{x_n\} \subset X \text{ such that } x_n \longrightarrow x \text{ and } T^{k_n}x_n \longrightarrow y\}.$$

denotes the *extended limit set of  $x$  under  $T$* .

An operator  $T$  is called a  $\mathcal{J}$ -class operator if there exists a non-zero vector  $z \in X$  such that  $\mathcal{J}(z) = X$ . It is evident that for a vector  $x \in X$

and an operator  $T \in B(X)$ , the set  $\mathcal{J}(x)$  is  $T$ -invariant. Closeness and some other topological properties associated with  $\mathcal{J}(x)$  can be found in [?]. Bellow, the concept of topologically transitive is introduced locally in the vector  $x \in X$ .

**Definition 2.3.** Let  $T$  be an operator. For every  $x \in M$ , where  $M$  is a subspace of  $X$ , the set  $\mathcal{LO}_M^{tr}(x)$  is;

$\mathcal{LO}_M^{tr}(x) = \{y \in X : \text{for every pair of neighborhoods } U, V \text{ of vectors } x, y \text{ respectively, there exists a positive integer } n, \text{ such that } T^n(U) \cap V \neq \emptyset \text{ and } T^n(M) \subseteq M\}.$

In what follows, if  $X = M$  and there is no ambiguity, then we write  $\mathcal{LO}^{tr}(x)$  instead of  $\mathcal{LO}_M^{tr}(x)$ , indeed we omit index  $M$ . Observe now that for every vector  $x \in X$ , the set  $Orb(T, x)$  is a subset of  $\mathcal{LO}^{tr}(x)$ . Obviously  $\mathcal{J}(x) \subseteq \mathcal{LO}^{tr}(x)$  but the following example shows that they are not necessarily equivalent.

**Example 2.4.** Consider the operator  $T = \lambda B$  where  $B$  is the backward shift operator on  $\ell^2(\mathcal{N})$ , the space of square summable sequences and  $\lambda \in (0, 1)$ . Let  $x \in \ell^2(\mathcal{N})$  be a vector such that  $Tx \neq 0$ . On the other hand for every strictly increasing sequence of positive integers  $\{k_n\}$  and every sequence  $\{x_n\} \subset \ell^2(\mathcal{N})$ , if  $x_n \rightarrow x$  then  $T^{k_n}x_n \rightarrow 0$ . Indeed  $\mathcal{J}(x) = \{0\}$  and therefore  $\mathcal{J}(x) \neq \mathcal{LO}^{tr}(x)$ , because the vector  $Tx \in \mathcal{LO}^{tr}(x)$ .

The following theorem shows under which additional assumption, two sets  $\mathcal{J}(x)$  and  $\mathcal{LO}^{tr}(x)$  are equivalent. Note that this equivalence helps us better understand how the concept of hypercyclicity was localized?

**Theorem 2.5.** Let  $x \in X$  and there exists  $\varepsilon > 0$  such that  $z \in \mathcal{J}(x)$  for every  $z \in B(x, \varepsilon)$ , then  $\mathcal{J}(x) = \mathcal{LO}^{tr}(x)$ .

*Proof.* we show that  $\mathcal{LO}^{tr}(x) \subseteq \mathcal{J}(x)$ . Consider  $y \in \mathcal{LO}^{tr}(x)$ ,  $\varepsilon < 1$  and smallest positive integers  $k_n$  such that

$$T^{k_n}B(x, \frac{\varepsilon}{n}) \cap B(y, \frac{1}{n}) \neq \emptyset.$$

There exists a positive integer  $m \geq k_n$  such that

$$B(x, \frac{\varepsilon}{n+1}) \cap T^{-m}B(y, \frac{1}{n+1}) \neq \emptyset,$$

and by continuity of  $T$  there exists a nonempty open set

$$W_{n+1} \subset B(x, \frac{\varepsilon}{n+1}) \cap T^{-m}B(y, \frac{1}{n+1}). \quad (2.1)$$

Now if  $m = k_n$ , then consider an  $x' \in W_{n+1}$  and since

$$x' \in W_{n+1} \subseteq \mathcal{J}(x') \subseteq \mathcal{LO}^{tr}(x'),$$

so it follows that there exists an integer  $n_0 > 0$  such that;

$$W'_{n+1} := T^{-n_0}W_{n+1} \cap W_{n+1} \quad (2.2)$$

is a nonempty open set. Thus

$$T^{n_0+m}(W'_{n+1}) \subset T^m(W_{n+1}) \subset B(y, \frac{1}{n+1}). \quad (2.3)$$

On the other hand by ?? and ?? we have  $W'_{n+1} \subset B(x, \frac{\varepsilon}{n+1})$ . Consequently  $T^{n_0+m}B(x, \frac{\varepsilon}{n+1}) \cap B(y, \frac{1}{n+1}) \neq \emptyset$ . Now set  $k_{n+1} := m + n_0 > k_n$  and take  $x_{n+1} \in B(x, \frac{\varepsilon}{n+1})$ . Therefore we find a strictly increasing sequence  $\{k_n\}$  of positive integers such that  $x_n \rightarrow x$  and  $T^{k_n}x_n \rightarrow y$ , in deed  $y \in \mathcal{J}(x)$ .  $\square$

**Question 2.6.** Let  $T$  be an operator on  $X$ . If  $M_1, M_2$  are two subspaces of  $X$  and for every vector  $x \in X$ ,  $\mathcal{LO}^{tr}_{M_i}(x) = M_i$  and  $i = 1, 2$ , is there any relation between  $M_1$  and  $M_2$ ?

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