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ON φ-AMENABILITY OF BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be an arbitrary Banach algebra and φ a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$. Our first purpose in this talk is to give some equivalent conditions under which guarantees a φ-mean of norm one. We deal with the problem of when $\|aa_\alpha - \varphi(a)a_\alpha\| \longrightarrow 0$ uniformly for all a in weakly compact subset of $\mathcal{A}$. We show that Banach algebras associated to a locally compact group G is responsive to this concept. Other results in this direction are also obtained.

1. Introduction

B. E. Johnson proved that a locally compact group G is amenable as a group if and only if the group algebra $L^1(G)$ is amenable as a Banach algebra. This result of Johnson laid the groundwork for amenability in Banach algebras. After the pioneering work of Johnson, several modifications of the original notion of amenability in Banach algebras are introduced. One of the most important modifications was introduced by A.T. Lau where he introduced the notion of left amenability for a class of F-algebras. This was latter generalized by E. Kaniuth in joint papers with A.T. Lau and J. Pym [5] where they introduced the notion of φ-amenability of Banach algebras.

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Recall that the second dual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ will always be equipped with the first Arens product which is defined as follows. For $a, b \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$, the elements fa and $m.f$ of $\mathcal{A}^*$ and $m.n \in \mathcal{A}^{**}$ are defined by $\langle f.a, b\rangle = \langle f, ab\rangle$, $\langle m.f, a\rangle = \langle m, f.a\rangle$ and $\langle m.n, f\rangle = \langle m, n.f\rangle$, respectively [1]. With this multiplication, $\mathcal{A}^{**}$ is a Banach algebra and $\mathcal{A}$ is a subalgebra of $\mathcal{A}^{**}$. Let $\mathcal{A}$ be an arbitrary Banach algebra. Let $\Delta(\mathcal{A})$ be the set of all non-zero characters, bounded multiplicative linear functionals on Banach algebra $\mathcal{A}$. For $\varphi \in \Delta(\mathcal{A})$, we call $\mathcal{A}$ is φ-amenable if there exists a bounded linear functional m on $\mathcal{A}^*$ satisfying $\langle m, \varphi\rangle = 1$ and $\langle m, f.a\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. For more details on φ-amenability of a Banach algebra the interested reader is referred to [2], [3] and [7]. Kaniuth, Lau and Pym in [4] characterized φ-amenability of $\mathcal{A}$ in terms of vanishing cohomology groups $H^1(\mathcal{A}, X^*)$ for a particular class of Banach $\mathcal{A}$-bimodules, and in terms of the existence of a bounded net $(a_\alpha)_\alpha$ in $\mathcal{A}$ satisfying $\varphi(a_\alpha) = 1$ for each α and $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ for each $a \in \mathcal{A}$. Various necessary and sufficient conditions of a global and a point-wise nature was found for a Banach algebra to possess a φ-mean of norm 1. The next result gives an important property that characterizes φ-amenability of $\mathcal{A}$, see Theorem 1.1 in [4].

Theorem 1.1. *Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta(\mathcal{A})$. Then the following conditions are equivalent:*

(a) *$\mathcal{A}$ is φ-amenable;*

(b) *There exists a net $(a_\alpha)_\alpha$ in $\mathcal{A}$ such that $\varphi(a_\alpha) = 1$ for all α and $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ for all $a \in \mathcal{A}$;*

(c) *If X is a Banach $\mathcal{A}$-bimodule such that $ax = \varphi(a)x$ for all $x \in X$ and $a \in \mathcal{A}$, then $H^1(\mathcal{A}, X^*) = \{0\}$.*

In this talk, we continue the study of φ-amenable Banach algebras. We present a few results in the theory of φ-amenable Banach algebras, and we obtain necessary and sufficient conditions for $\mathcal{A}$ to have a φ-mean. Let $\mathcal{A}$ be a φ-amenable Banach algebra. There exists a net $(a_\alpha)_\alpha$ in $\mathcal{A}$ such that $\varphi(a_\alpha) = 1$ for all α and $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ for all $a \in \mathcal{A}$. It remains an open question, to the author's knowledge, when $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ uniformly for all a in weakly compact subsets of $\mathcal{A}$. We shall investigate that this problem is true over a group algebra.

2. Main reasults

In the following Theorem we establish several criteria for $\mathcal{A}$ to possess a φ-mean of norm 1.

Theorem 2.1. *Let $\mathcal{A}$ be any Banach algebra and $\varphi \in \Delta(\mathcal{A})$. Let H denote the real-linear span of the set $\{\varphi(a)f.b - \varphi(b)f.a; a, b \in \mathcal{A}, f \in \mathcal{A}^*\}$. Then the following conditions are equivalent:*

(a) *There exists a φ-mean m with $\|m\| = 1$;*

(b) *For every $\epsilon > 0$ and $h \in H$,*

$$\sup\{Re\langle h, a\rangle; a \in \mathcal{A}, \varphi(a) = 1, \|a\| < 1 + \epsilon\} > 0.$$

The next theorem, which is one of the main results, in particular shows that the existence of a φ-mean of norm 1 is a point-wise property in the sense that it follows from the existence of a certain functional on $\mathcal{A}^*$ associated with each of the elements of $\mathcal{A}^*$ and $\mathcal{A}$.

Theorem 2.2. *Let $\mathcal{A}$ be any Banach algebra and $\varphi \in \Delta(\mathcal{A})$. Then the following conditions are equivalent:*

(a) *$\mathcal{A}$ admits a φ-mean of norm 1;*

(b) *For every $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$, there exists a mean $m_{f,a}$ on $\mathcal{A}^*$ such that $\langle m_{f,a}, \varphi\rangle = 1$, $\|m_{f,a}\| = 1$ and $\langle \varphi(b)m_{f,a}, f.ab\rangle = \varphi(ab)\langle m_{f,a}, f.b\rangle$ whenever $b \in \mathcal{A}$;*

(c) *There exists $m \in \mathcal{A}^{**}$ such that $\|m\| = \langle m, \varphi\rangle = 1$ and $\langle m, f.a\rangle = \langle m, f.b\rangle$ for all $f \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$ with $\varphi(a) = \varphi(b) = 1$.*

Let G be a locally compact group and let $M_\circ(G)$ be the set of all probability measures in $M(G)$. It is known that [8], G is amenable if and only if there is a net $(\mu_\alpha)_\alpha$ in $M_\circ(G)$ such that for all compact subset K of G, $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly over all μ in $M_\circ(G)$ which are supported in K. For a semitoplogical semigroup S this fact is not known. The first author in [3] proved that if $M_\circ(S)$ has a measure μ such that the map $s \mapsto \delta_s * \mu$ from S into $M(S)$ is continuous, then the last statement is true. In fact with this condition he provide an answer to a problem raised by Lau [6]. Let $\mathcal{A}$ be φ-amenable. It is easy to see that there exists a net $(a_\alpha)_\alpha$ in $\mathcal{A}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ uniformly for all a in compact subsets of $\mathcal{A}$. It is an open question when $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ uniformly for all a in weakly compact subsets of $\mathcal{A}$.

Theorem 2.3. *Let G be a locally compact topological group and $a \in L^1(G)$. Then the following conditions are equivalent:*

(a) *$L^1(G)$ admits a φ-mean of norm 1;*

(b) *There exists a net $(a_\alpha)_\alpha$ in $\{a \in L^1(G); \|a\|_1 = 1\}$ such that $\varphi(a_\alpha) \to 1$ and also for each weakly compact subset $\mathbf{C} \subseteq M_\circ(G) \cap L^1(G)$, $(\varphi(a)ba_\alpha - \varphi(b)aa_\alpha) \to 0$ uniformly for all $a \in \mathbf{C}$;*

(c) *There is a net* $(a_\alpha)_\alpha \in M_\circ(G)\cap L^1(G)$ *such that for each weakly compact subset* $\mathbf{C} \subseteq M_\circ(G) \cap L^1(G)$, $\|aa_\alpha - \varphi(a)a_\alpha\| \longrightarrow 0$ *uniformly for all* $a \in \mathbf{C}$.

Theorem 2.4. *Let* $\mathcal{A}$ *be a Banach algebra and* $\varphi \in \Delta(\mathcal{A})$. *Then the following statements are equivalent:*

(a) *There exists a net* $(a_\alpha)_\alpha$ *in* $\{a \in \mathcal{A}; \varphi(a) = 1\}$ *such that* $(a_\alpha)_\alpha$ *converges to some left invariant* φ*-mean* m *with* $\|m\| = 1$ *in the weak* topology and* $(a_\alpha)_\alpha$ *converges strongly to a left invariance uniformly on weakly compact subsets of* $\mathcal{A}$*;*

(b) *For every weakly compact subset* $\mathbf{C}$ *of* $\mathcal{A}$ *and* $\epsilon > 0$,

$$\inf\{\sup\{\|ca\|;\ c \in \mathbf{C}\},\ \varphi(a) = 1,\ \|a\| \leq 1+\epsilon\} \leq (1+\epsilon)\sup\{|\varphi(c)|;\ c \in \mathbf{C}\};$$

(c) *There exists a net* $(a_\alpha)_\alpha$ *in* $\mathcal{A}$ *such that* $\varphi(a_\alpha) = 1$ *for all* α, $\|a_\alpha\| \to 1$ *and* $\lim_\alpha \|aa_\alpha\| = |\varphi(a)|$ *uniformly on weakly compact subsets of* $\mathcal{A}$.

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