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UNITIZATION OF DUAL BANACH ALGEBRAS AND THEIR CONNES AMENABILITY

EBRAHIM TAMIMI *

Department of Mathematics, University of Semnan
tamimi_ebrahim@semnan.ac.ir

Abstract. Let φ be a ω^*-continuous homomorphism from a dual Banach algebra to $\mathbb{C}$. In this paper, first we investigate the Connes amenability of unitization of φ, then we define a type of diagonal for certain dual Banach algeras and prove that the existence of such a diagonal is equivalent to unitization of Connes amenability.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and E be a Banach $\mathcal{A}$-bimodule. A continuous Linear operator $D : \mathcal{A} \to E$ is a derivation if it satisfies $D(ab) = D(a).b + a.D(b)$ for all $a, b \in \mathcal{A}$. Given $x \in E$, the derivation $ad_x : \mathcal{A} \to E$ is defined by $ad_x(a) = a.x - x.a$, is called inner derivation. In [3], the amenability for Banach algebras is introduced by johnson. A Banach algebra $\mathcal{A}$ is amenable if for every Banach $\mathcal{A}$-bimodule E, every derivation from $\mathcal{A}$ into E^*, the dual of E, is inner. We know that the projective tensor product $\mathcal{A}$ with itself is a Banach $\mathcal{A}$-bimodule. So, the map $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ defined by $\pi(a \otimes b) = ab$ is an $\mathcal{A}$-bimodule homomorphism.
Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is dual if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We say E_*

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* Speaker.

predual of E . A dual Banach $\mathcal{A}$-bimodule E is normal if the module actions of $\mathcal{A}$ on E are ω^*-continuous. A Banach algebra is dual if it is dual as a Banach $\mathcal{A}$-bimodule. If we want to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$, we write $\mathcal{A} = (\mathcal{A}_*)^*$ Connes amenability, which seems to be a natural variant of amenability for dual Banach algebras, was introduced by V. Runde [6]. A dual Banach algebra $\mathcal{A}$ is Connes amenable if every ω^*-continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule, then $\sigma wc(E)$ stands for the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \to E, a \to \{^{x.a}_{a.x}$$

are ω^*-ω continuous. It is a closed submodule of E.
A generalization of amenability which depends on homomorphisms was introduced by E. Kaniuth, A. T. Lau and J. Pym in [4]. This concept was also studied independently by M. S. Monfared in [5]. Let $\mathcal{A}$ be a Banach algebra and φ be a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$. We say $\mathcal{A}$ is φ-amenable if there exists a bounded linear functional m on $\mathcal{A}^*$ satisfying $m(\varphi) = 1$ and $m(f.a) = \varphi(a)m(f)$, for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. We write $\Delta(\mathcal{A})$ for the set of all homomorphism from $\mathcal{A}$ onto $\mathbb{C}$. Let $\mathcal{A}$ be a dual Banach algebra. It is known that its unitization $\mathcal{A}^\sharp = \mathcal{A} \oplus \mathbb{C}e$, is a dual Banach algebra as well, where e is the identity of $\mathcal{A}^\sharp$. Let $\varphi \in \Delta_{\omega^*}(\mathcal{A})$ and let $\varphi^\sharp$ be its unique extension to $\mathcal{A}^\sharp$, i.e. $\varphi^\sharp(a + \lambda e) = \varphi(a) + \lambda$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$. It is obvious that $\varphi^\sharp \in \Delta_{\omega^*}(\mathcal{A}^\sharp)$. In this note we study $\varphi^\sharp$-Connes amenability for dual Banach algebras. Thus, we study basic properties of $\varphi^\sharp$-Connes amenability. We characterize it through vanishing of $\mathbf{H}^1_{\omega^*}(\mathcal{A}^\sharp, E)$ for certain Banach $\mathcal{A}^\sharp$-bimodule E. A number of hereditary properties are also discussed. Also, we define a type of virtual diagonal for a dual Banach algebra $\mathcal{A}^\sharp$, showing that the existence of such a diagonal is equivalent to $\varphi^\sharp$-Connes amenability of $\mathcal{A}^\sharp$.

2. Main results

Suppose that $\mathcal{A}$ is a dual Banach algebra and φ is a homomorphism frome $\mathcal{A}$ onto $\mathbb{C}$. Then it is an easy observation that φ is ω^*-continuous if and only if $\varphi \in \sigma wc(\mathcal{A}^*)$. For a dual Banach algebra $\mathcal{A}$, $\Delta_{\omega^*}(\mathcal{A})$ will denote the set of all ω^*-continuous homomorphisms from $\mathcal{A}$ onto $\mathbb{C}$.
Let $\mathcal{A}$ be a dual Banach algebra. It is known that its unitization $\mathcal{A}^\sharp = \mathcal{A} \oplus \mathbb{C}e$, is a dual Banach algebra as well, where e is the identity of $\mathcal{A}^\sharp$. We define $f_0 : \mathcal{A}^\sharp \to \mathbb{C}$ by $f_0(e) = 1$ and $f_0 \mid_{\mathcal{A}} = 0$ so that $(\mathcal{A}^\sharp)^* = \mathcal{A}^* \oplus \mathbb{C}f_0$.
Let $\varphi \in \Delta_{\omega^*}(\mathcal{A})$ and let $\varphi^\sharp$ be its unique extension to $\mathcal{A}^\sharp$, i.e., $\varphi^\sharp(a +$

$\lambda e) = \varphi(a) + \lambda$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$. It is obvious that $\varphi^\sharp \in \Delta_{\omega^*}(\mathcal{A}^\sharp)$. For the right module action in f_0, we have $f_0.(a + \lambda e) = \lambda f_0$, since $f_0.a = 0$, for all $a \in \mathcal{A}$. We may identify $(\mathbb{C}f_0)^*$ with $\mathbb{C}m_0$, where m_0 is a functional on $(\mathcal{A}^\sharp)^*$ defined by $m_0(f_0) = 1$ and $m_0 \mid_{\mathcal{A}^*} = 0$. Therefore, if we consider $\mathbb{C}f_0$ as a sub $\mathcal{A}^\sharp$-bimodule of $(\mathcal{A}^\sharp)^*$, then we see that $f_0 \in \sigma wc(\mathbb{C}f_0)$ so that $\sigma wc(\mathbb{C}f_0) = \mathbb{C}f_0$. Therefore, we conclude that $\sigma wc((\mathcal{A}^\sharp)^*) = \sigma wc(\mathcal{A}^*) \oplus \mathbb{C}f_0$, so that $\sigma wc((\mathcal{A}^\sharp)^*)^* = \sigma wc(\mathcal{A}^*)^* \oplus \mathbb{C}m_0$.

Definition 2.1. Suppose that $\mathcal{A}$ is a dual Banach algebra and $\varphi \in \Delta_{\omega^*}(\mathcal{A})$. We call $\mathcal{A}$ φ-Connes amenable if $\mathcal{A}$ admits a φ-Connes mean m, i.e. there exists a bounded linear functional m on $\sigma wc(\mathcal{A}^*)$ satisfying $m(\varphi) = 1$ and $m(f.a) = \varphi(a)m(f)$ for all $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$.

We recall some terminology from [6] and [2]. Let $\mathcal{A}$ be a dual Banach algebra and E be a normal, dual Banach $\mathcal{A}$-bimodule. We write $\mathbf{Z}^1_{\omega^*}(\mathcal{A}, E)$ for the set of all ω^*-continuous derivations from $\mathcal{A}$ to E. Clearly $B^1(\mathcal{A}, E)$ the set of all inner derivations from $\mathcal{A}$ to E, is a subspace of $\mathbf{Z}^1_{\omega^*}(\mathcal{A}, E)$. Whence, we have the meaningful definition $\mathbf{H}^1_{\omega^*}(\mathcal{A}, E) = \frac{\mathbf{Z}^1_{\omega^*}(\mathcal{A}, E)}{B^1(\mathcal{A}, E)}$.

Theorem 2.2. *suppose that $\mathcal{A}$ is a dual Banach algebra, its unitization $\mathcal{A}^\sharp = \mathcal{A} \oplus \mathbb{C}e$, is a dual Banach algebra as well and $\varphi^\sharp \in \Delta_{\omega^*}(\mathcal{A}^\sharp)$. Then the following conditions are equivalent:*
(i) $\mathcal{A}^\sharp$ has $\varphi^\sharp$-Connes mean;
(ii) if $E = (E_*)^*$ is a normal, dual Banach $\mathcal{A}^\sharp$-bimodule such that $x.(a + \lambda e) = \varphi^\sharp(a + \lambda e)x$ for all $x \in E$ and $a + \lambda e \in \mathcal{A}^\sharp$, then $\mathbf{H}^1_{\omega^*}(\mathcal{A}^\sharp, E) = \{0\}$.

Theorem 2.3. *Let $\mathcal{A}$ be a dual Arens regular Banach algebra and $\varphi \in \Delta_{\omega^*}(\mathcal{A})$. Then $\mathcal{A}^{**}$ is φ^{**}-Connes amenable if and only if $\mathcal{A}^\sharp$ is $\varphi^\sharp$-Connes amenable.*

Theorem 2.4. *Suppose that $\mathcal{A}^\sharp$ and $\mathcal{B}^\sharp$ are dual Banach algebras, $\theta : \mathcal{A}^\sharp \longrightarrow \mathcal{B}^\sharp$ is a continuous and ω^*-continuous homomorphism with ω^*-dense range, and that $\varphi^\sharp \in \Delta_{\omega^*}(\mathcal{B}^\sharp)$. If $\mathcal{A}^\sharp$ is $\varphi^\sharp o\theta$-Connes amenable, then $\mathcal{B}^\sharp$ has $\varphi^\sharp$-Connes mean.*

Proof. Consider the diagram

$$\begin{tikzcd} & \mathbb{C} & \\ \mathcal{A}^\sharp \arrow[ur, "\varphi^\sharp o\theta"] \arrow[rr, "\theta"] & & \mathcal{B}^\sharp \arrow[ul, "\varphi^\sharp"'] \end{tikzcd}$$

Notice that $\varphi^\sharp o\theta \in \Delta_{\omega^*}(\mathcal{A}^\sharp)$. Suppose that $m \in \sigma wc((\mathcal{A}^\sharp)^*)^*$ satisfies $m(\varphi^\sharp o\theta) = 1$ and $m(f.a) = (\varphi^\sharp o\theta)(a)m(f)$ for all $(a + \lambda e) \in \mathcal{A}^\sharp$ and

$f \in \sigma wc((\mathcal{A}^\sharp)^*)$. Define $n \in \sigma wc((\mathcal{B}^\sharp)^*)^*$ by $n(g) = m(go\theta)$ for $g \in \sigma wc((\mathcal{B}^\sharp)^*)$. Next, for $(a+\lambda e) \in \mathcal{A}^\sharp$ and $g \in \sigma wc((\mathcal{B}^\sharp)^*)$ we have $(g.\theta(a+\lambda e))o\theta = (go\theta).(a+\lambda e)$ and hence

$$n(g.\theta(a+\lambda e)) = m((g.\theta(a+\lambda e))o\theta) = m((go\theta).(a+\lambda e))$$

$$= (\varphi^\sharp o\theta)(a+\lambda e)m(go\theta) = (\varphi^\sharp o\theta)(a+\lambda e)n(g)$$

Since $\theta(\mathcal{A}^\sharp)$ is ω^*-dense in $\mathcal{B}^\sharp$, the above equation suffices to prove $\varphi^\sharp$-Connes amenability of $\mathcal{B}^\sharp$. □

Analogously, we may obtain the following:

Corollary 2.5. *Suppose that $\mathcal{A}^\sharp$ is Banach algebra, $\mathcal{B}^\sharp$ is a dual Banach algebra, $\theta : \mathcal{A}^\sharp \longrightarrow \mathcal{B}^\sharp$ is a continuous homomorphism with ω^*-dense range, and that $\varphi^\sharp \in \Delta_{\omega^*}(\mathcal{B}^\sharp)$. If $\mathcal{A}^\sharp$ is $\varphi^\sharp o\theta$-amenable, then $\mathcal{B}^\sharp$ has $\varphi^\sharp$-Connes mean.*

Theorem 2.6. *Let $\mathcal{A}^\sharp$ be a Arens regular Banach algebra which is an ideal in $(\mathcal{A}^\sharp)^{**}$, and let $\varphi^\sharp \in \Delta(\mathcal{A}^\sharp)$. Let $\widetilde{\varphi^\sharp}$, the extension of $\varphi^\sharp$ to $(\mathcal{A}^\sharp)^{**}$, belongs to $\Delta_{\omega^*}(\mathcal{A}^\sharp)^{**}$. Then the following conditions are equivalent:*
(i) $\mathcal{A}^\sharp$ is $\varphi^\sharp$-amenable;
(ii) $(\mathcal{A}^\sharp)^{**}$ is $\widetilde{\varphi^\sharp}$-Connes amenable.

Proof. $(i) \Longrightarrow (ii)$ Because $\varphi^\sharp = \widetilde{\varphi^\sharp}o\iota$, where $\iota : \mathcal{A}^\sharp \hookrightarrow (\mathcal{A}^\sharp)^{**}$ is the inclusion map, this is an immediate consequence of Corollary 2.5.
$(ii) \Longrightarrow (i)$ By the assumption, there is $m \in \sigma wc((\mathcal{A}^\sharp)^{***})^*$ such that $m(\widetilde{\varphi^\sharp}) = 1$ and $m(F.u) = \widetilde{\varphi^\sharp}(u)m(F)$, for $u \in (\mathcal{A}^\sharp)^{**}$ and $F \in \sigma wc((\mathcal{A}^\sharp)^{***})$. Set $\overline{m} = m \mid_{(\mathcal{A}^\sharp)^*}$, the restriction of m to $(\mathcal{A}^\sharp)^*$. Since $(\mathcal{A}^\sharp)^{**}$ is a dual Banach agebra, $(\mathcal{A}^\sharp)^* \subseteq \sigma wc((\mathcal{A}^\sharp)^{***})$ and therefore $\overline{m}$ is well-difined. Then, it is readily seen that $\overline{m}(\varphi^\sharp) = m(\widetilde{\varphi^\sharp}) = 1$ and $\overline{m}(f.(a+\lambda e)) = \varphi^\sharp(a+\lambda e)\overline{m}(f)$, $(a+\lambda e) \in \mathcal{A}^\sharp$, $f \in (\mathcal{A}^\sharp)^*$.
□

Remark 2.7. Let $\mathcal{A}^\sharp = ((A^\sharp)_*)^*$ be a dual Banach algebra. We can see that $\pi^*((A^\sharp)_*) \subseteq \sigma wc((\mathcal{A}^\sharp\widehat{\otimes}\mathcal{A}^\sharp)^*)$ and then taking adjoint, also we can extend π to an $\mathcal{A}^\sharp$-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc((\mathcal{A}^\sharp\widehat{\otimes}\mathcal{A}^\sharp)^*)^*$ to $\mathcal{A}^\sharp$.

Definition 2.8. Let $\mathcal{A}^\sharp = ((A^\sharp)_*)^*$ be a dual Banach algebra. A σwc-virtual diagonal for $\mathcal{A}^\sharp$ is an element $M \in \sigma wc((\mathcal{A}^\sharp \otimes \mathcal{A}^\sharp)^*)^*$ such that for all $(a+\lambda e) \in \mathcal{A}^\sharp$ we have
(i) $(a+\lambda e).M = M.(a+\lambda e)$;
(ii) $(a+\lambda e)\pi_{\sigma wc}(M) = (a+\lambda e)$.

Corollary 2.9. *Let $\mathcal{A}^\sharp = ((\mathcal{A}^\sharp)_*)^*$ be a dual Banach algebra. Then Connes amenability of $\mathcal{A}^\sharp$ is equivalent to existence of a σwc-virtual diagonal for $\mathcal{A}^\sharp$.*

By using of [8], we conclude that

$$\pi^*(\sigma wc((\mathcal{A}^\sharp)^*)) \subseteq \sigma wc((\mathcal{A}^\sharp \widehat{\otimes} \mathcal{A}^\sharp)^*).$$

So if $\varphi^\sharp \in \Delta_{\omega^*}(\mathcal{A}^\sharp)$, then

$$\varphi^\sharp \otimes \varphi^\sharp = \pi^*(\varphi^\sharp) \in \sigma wc((\mathcal{A}^\sharp \widehat{\otimes} \mathcal{A}^\sharp)^*),$$

where

$$\varphi^\sharp \otimes \varphi^\sharp((a+\lambda e) \otimes (b+\lambda e)) = \varphi^\sharp(a+\lambda e)\varphi^\sharp(b+\lambda e), \qquad a+\lambda e, b+\lambda e \in \mathcal{A}^\sharp.$$

Thus we obtain the following definition.

Definition 2.10. Let $\mathcal{A}^\sharp$ be a dual Banach algebra, and let $\varphi^\sharp \in \Delta_{\omega^*}(\mathcal{A}^\sharp)$. An element $M \in \sigma wc((\mathcal{A}^\sharp \widehat{\otimes} \mathcal{A}^\sharp)^*)^*$ is a $\varphi^\sharp$-σwc virtual diagonal for $\mathcal{A}^\sharp$ if
(*i*) $\langle \varphi^\sharp \otimes \varphi^\sharp, M \rangle = 1$;
(*ii*) $(a+\lambda e).M = \varphi^\sharp(a+\lambda e).M, \quad (a+\lambda e) \in \mathcal{A}^\sharp$.

Remark 2.11. Let $\mathcal{A}^\sharp$ be a dual Banach algebra. Taking adjoint of the restriction map $\pi^* \mid_{\sigma wc(\mathcal{A}^\sharp)^*}$, we obtain an $\mathcal{A}^\sharp$-bimodule homomorphism;

$$\pi^0_{\sigma wc} : \sigma wc((\mathcal{A}^\sharp \widehat{\otimes} \mathcal{A}^\sharp)^*)^* \longrightarrow \sigma wc((A^\sharp)^*)^*.$$

Because we choose homomorphisms from $\sigma wc(A^\sharp)^*$, which is larger than $(A^\sharp)_*$, working with $\pi^0_{\sigma wc}$ seems more natural than that of $\pi_{\sigma wc}$. As a consequence, we observe that

$$\langle \varphi^\sharp \otimes \varphi^\sharp, M \rangle = \langle \varphi^\sharp, \pi^0_{\sigma wc}(M) \rangle$$

whenever $\varphi^\sharp \in \Delta_{\omega^*}(\mathcal{A}^\sharp)$ and $M \in \sigma wc((\mathcal{A}^\sharp \widehat{\otimes} \mathcal{A}^\sharp)^*)^*$.

With these preparations, we can now characterize $\varphi^\sharp$-Connes amenable dual Banach algebras through the existence of $\varphi^\sharp$-σwc virtual diagonals.

From [1], we conclude the following result:

Theorem 2.12. *Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A})$. Then the following conditions are equivalent:*
(*i*) *$\mathcal{A}^\sharp$ has $\varphi^\sharp$-Connes mean;*
(*ii*) *There is a φ-σwc virtual diagonal for $\mathcal{A}$;*
(*iii*) *There is a $\varphi^\sharp$-σwc virtual diagonal for $\mathcal{A}^\sharp$.*

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