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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A SINGULAR (p, q)-LAPLACIAN PROBLEM

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Abstract. In this article, Using the variational method and critical point theory, we prove the existence two weak solutions for a (p, q)-Laplacian boundary value problem with singular nonlinearities in a smooth bounded domain in $\mathbb{R}^N$.

1. Introduction

The quasilinear operator (p, q)-Laplacian has been used to model steady-state solutions such as reaction-diffusion problems. The differential operator $\Delta_p + \Delta_q$ is known as the (p, q)-Laplacian operator, if $p \neq q$.

We point out that in [1] the existence weak solutions for the quasilinear elliptic problem of sigaular (p, q)-Laplacian is studied and in [3, 4] the authors proved the existence of nontrivial weak solutions of elliptic equations.

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Here, we consider the singular (p, q)-Laplacian elliptic problem as

$$\begin{cases} -\Delta_p u - \Delta_q u + \dfrac{|u|^{p-2}u}{|x|^p} + \dfrac{|u|^{q-2}u}{|x|^q} = \lambda f(x, u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $2 \leq q < p < N$, $\lambda > 0$ is a real parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carath*é*odory function such that

(f_1) $|f(x, t)| \leq a_1 t + a_2 |t|^{r-1}$, for all $(x, t) \in \Omega \times \mathbb{R}$,

where a_1 and a_2 are positive constants, $r \in]p, p^*[$. Note that $p^* = \dfrac{Np}{N-p}$ is the critical Sobolev exponent.

Let $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \|u\|_p = (\int_\Omega |\nabla u(x)|^p dx)^{\frac{1}{p}}, \tag{1.2}$$

and the norm in $L^p(\Omega)$ is

$$|u|_p = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}}. \tag{1.3}$$

Assume $r \in [1, p^*[$, the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ shows that there exists a $c_r > 0$ such that

$$\|u\|_{L^r(\Omega)} \leq c_r \|u\|_p \qquad \text{for all } u \in W_0^{1,p}(\Omega), \tag{1.4}$$

where c_r is the best constant of the embedding.

We recall the classical Hardy's inequality:

$$\int_\Omega \frac{|u(x)|^s}{|x|^s} dx \leq \frac{1}{H} \int_\Omega |\nabla u(x)|^s dx, \qquad \text{for all } u \in W_0^{1,s}(\Omega), \tag{1.5}$$

where $1 < s < N$ and $H := (\frac{N-s}{s})^s$, see [5].

If we set $F(x, \xi) := \int_0^\xi f(x, t) dt$, for every $(x, \xi) \in \Omega \times \mathbb{R}$, then the energy functional $I_\lambda : X \to \mathbb{R}$ associated with (1.1) can be written

$$I_\lambda := \Phi(u) - \lambda \Psi(u), \qquad \text{for all } u \in X,$$

where

$$\Phi(u) := \Phi_p(u) + \Phi_q(u),$$

such that

$$\Phi_p(u) := \frac{1}{p}\Big(\int_\Omega |\nabla u|^p dx + \int_\Omega \frac{|u|^p}{|x|^p} dx\Big)$$
$$\Phi_q(u) := \frac{1}{q}\Big(\int_\Omega |\nabla u|^q dx + \int_\Omega \frac{|u|^q}{|x|^q} dx\Big)$$
$$\Psi(u) := \int_\Omega F(x,u(x))dx.$$

By (1.5),

$$\frac{\|u\|^p}{p} \le \Phi_p(u) \le \Big(\frac{H+1}{pH}\Big)\|u\|^p, \quad \frac{\|u\|^q}{q} \le \Phi_q(u) \le \Big(\frac{H+1}{qH}\Big)\|u\|^q, \tag{1.6}$$

for every $u \in X$.

Definition 1.1. The function $u : \Omega \to \mathbb{R}$ is a weak solution of (1.1), if $u \in X$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^p} uv dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx + \int_\Omega \frac{|u|^{q-2}}{|x|^q} uv dx - \lambda \int_\Omega f(x,u)v dx = 0,$$

for every $v \in X$.

Since Ω is bounded and $q < p$, we have $W_0^{1,p}(\Omega) \subset W_0^{1,q}(\Omega)$ and the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$.

Definition 1.2. A Gâteaux differentiable function I satisfies the Palais-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that

(I) $\{I(u_n)\}$ is bounded,
(II) $\limsup_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

We need the following proposition and theorem to prove the main result.

Proposition 1.3. *The operator $T : X \to X^*$ defined by*

$$T(u)(v) := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^p} uv dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx + \int_\Omega \frac{|u|^{q-2}}{|x|^q} uv dx,$$

for every $u, v \in X$, is strictly monotone.

Theorem 1.4. [2, Theorem 3.2] *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $\delta > 0$ such that $sup_{\{\Phi(u)<\delta\}}\Psi(u) < +\infty$ and assume that, for each $\lambda \in \Lambda := \left]0, \frac{\delta}{sup_{\{\Phi(u)<\delta\}}\Psi(u)}\right[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below. Then, for each $\lambda \in \Lambda$ the functional I_λ admits two distinct critical points.*

2. Main results

The statement of main result is as follows:

Theorem 2.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that condition (f_1) holds. Moreover, assume that*

(f_2) *There exist $\theta > p$ and $K > 0$ such that*

$$0 < \theta F(x,t) \leq t f(x,t),$$

for each $x \in \Omega$ and $|t| > K$.

Then for each $\lambda \in]0, \lambda^*[$, problem (1.1) admits at least two distinct weak solutions, where

$$\lambda^* := \frac{r}{r a_1 c_1 p^{\frac{1}{p}} + a_2 c_r^r p^{\frac{r}{p}}}$$

and c_r is the constant of the embedding $X \hookrightarrow L^r(\Omega)$ for each $r \in [1, p^*[$ in (1.4).

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