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COMPLEX SYMMETRIC WEIGHTED COMPOSITION OPERATORS

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Abstract. In this paper, we find complex symmetric weighted composition operators with special conjugations.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space, denoted $H^2(\mathbb{D}) = H^2$, is the set of all analytic functions f on $\mathbb{D}$, satisfying the norm condition

$$\|f\|^2 = \lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

The space $H^\infty(\mathbb{D}) = H^\infty$ consists of all the functions that are analytic and bounded on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Let φ be an analytic map from the open unit disk $\mathbb{D}$ into itself. The operator that takes the analytic map f to $f \circ \varphi$ is a composition operator and is denoted by C_φ. A natural generalization of a composition operator is an operator that takes f to $\psi \cdot f \circ \varphi$, where ψ is a fixed analytic map on $\mathbb{D}$. This operator is aptly named a weighted composition operator and is usually denoted by $C_{\psi,\varphi}$. More precisely, if z is in the

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unit disk then $(C_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$. Throughout this paper, we consider $C_{\psi,\varphi}$ on H^2.

A bounded operator T on a complex Hilbert space H is said to be a complex symmetric operator if there exists a conjugation C (an isometric, antilinear and involution) such that $CT^*C = T$. The complex symmetric operators class was initially addressed by Garcia and Putinar (see [4] and [5]) and includes the normal operators, Hankel operators and Volterra integration operators. Invoking [6, Theorem 2], any composition operator with an involutive automorphism symbol is complex symmetric. In [1], Bourdon et al. showed that among the automorphisms of $\mathbb{D}$, only the elliptic ones may introduce complex symmetric operators. Moreover, they proved that for φ, not the rotation and involutive automorphism, which is elliptic automorphism of order q that $4 \leq q \leq \infty$, C_φ is not complex symmetric. In this paper we use the symbol J for the special conjugation that $(Jf)(z) = \overline{f(\overline{z})}$ for each analytic function f. In [3] and [7], all J-symmetric weighted composition operators were characterized. Recently in [8] Narayan et al. have found complex symmetric composition operators whose symbols are linear-fractional, but not an automorphism. In this paper, we investigate the results from [2]. We find all unitary weighted composition operators which are J-symmetric. Then we consider the special conjugations which are the products of these unitary weighted composition operators and the conjugation J. Next, we obtain complex symmetric weighted composition operators with these conjugations. In addition, we characterize all complex symmetric weighted composition operators which are isometries.

2. Main results

An operator T is said to be unitary if $T^*T = TT^* = I$. In the following proposition, we find all unitary weighted composition operators $C_{\psi,\varphi}$ which are J-symmetric.

Proposition 2.1. *The weighted composition operator $C_{\psi,\varphi}$ is unitary and J-symmetric if and only if either $\psi(z) = c\frac{(1-|p|^2)^{1/2}}{1-\overline{p}z}$ and $\varphi(z) = \frac{\overline{p}}{p}\frac{p-z}{1-\overline{p}z}$, where $p \in \mathbb{D} - \{0\}$ and $|c| = 1$ or $\psi \equiv \mu$ and $\varphi(z) = \lambda z$, when $|\mu| = |\lambda| = 1$.*

From now, we assume that $\varphi_p(z) = \frac{\overline{p}}{p}\frac{p-z}{1-\overline{p}z}$, where $p \in \mathbb{D} - \{0\}$ and $\psi_p(z) = c\frac{(1-|p|^2)^{1/2}}{1-\overline{p}z}$, when $p \in \mathbb{D}$ and $|c| = 1$.

Lemma 2.2. *If U is a unitary and complex symmetric operator with conjugation C, then UC is a conjugation.*

Proposition 2.3. *Suppose that U is unitary and complex symmetric with conjugation WJ, where W is unitary. Then an operator A is WJ-symmetric if and only if UA is UWJ-symmetric.*

In the following theorem, we find all complex symmetric weighted composition operators with conjugation UJ that U is unitary and J-symmetric weighted composition operator which was stated in Proposition 2.1.

Theorem 2.4. *Let $a_0 \in \mathbb{D}$ and $a_1, b \in \mathbb{C}$. Suppose that $\psi(z) = \frac{b}{1-a_0 z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$ that φ is an analytic self-map of $\mathbb{D}$.*
(1) For $p \neq 0$, the weighted composition operator $C_{\widetilde{\psi},\widetilde{\varphi}}$ is complex symmetric with conjugation $C_{\psi_p,\varphi_p}J$ if and only if $\widetilde{\psi} = \psi_p \cdot \psi \circ \varphi_p$ and $\widetilde{\varphi} = \varphi \circ \varphi_p$ for some φ and ψ.
(2) For $|\lambda| = 1$, the weighted composition operator $C_{\widetilde{\psi},\widetilde{\varphi}}$ is complex symmetric with conjugation $C_{\lambda z}J$ if and only if $\widetilde{\psi} = \psi(\lambda z)$ and $\widetilde{\varphi}(z) = \varphi(\lambda z)$ for some φ and ψ.

In Theorem 2.5, we show that a weighted composition operator which is both a complex symmetric operator and an isometry is unitary; moreover, we find all conjugations for unitary weighted composition operators.

Theorem 2.5. *A weighted composition operator $C_{\psi,\varphi}$ is both an isometry and a complex symmetric operator if and only if $\varphi(z) = \lambda \frac{p-z}{1-\overline{p}z}$ and $\psi \equiv \psi_p$, where $|\lambda| = 1$ and $p \in \mathbb{D}$. Furthermore, if $p \neq 0$, then the conjugation for $C_{\psi,\varphi}$ is $C_{\psi_p,\varphi_p}J$.*

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