The Extended Abstracts of
The 6th Seminar on Functional Analysis and its Applications
4-5th March 2020, University of Isfahan, Iran

NUMERICAL RANGES OF WEIGHTED COMPOSITION OPERATORS

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Abstract. In this paper, we investigate the numerical range of $C_{\psi,\varphi}$, when φ has a fixed point in $\mathbb{D}$.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and the Hardy space H^2 consisting of the functions $f(z) = \Sigma_{n=0}^{\infty}\hat{f}(n)z^n$ holomorphic in $\mathbb{D}$ such that $\|f\|^2 = \Sigma_{n=0}^{\infty}|\hat{f}(n)|^2 < \infty$ with $\hat{f}(n)$ denoting the n-th Taylor coefficient of f.

We recall that $H^\infty(\mathbb{D}) = H^\infty$ is the space of all bounded analytic functions defined on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$. Let φ be an analytic self-map of $\mathbb{D}$, then the equation $C_\varphi(f) = f \circ \varphi$ defines a composition operator C_φ with inducing map φ. For an analytic function ψ on $\mathbb{D}$ and an analytic self-map φ of $\mathbb{D}$, the weighted composition operator $C_{\psi,\varphi} : H^2 \to H^2$ is given by $C_{\psi,\varphi}h = \psi \cdot (h \circ \varphi)$. For $g \in L^\infty(\partial\mathbb{D})$, the Toeplitz operator T_g is the operator on H^2 given by $T_g(f) = P(gf)$ for f in H^2, where P is the orthogonal projection of L^2 onto H^2.

2010 *Mathematics Subject Classification.* Primary: 47B38; Secondary: 47B33.
Key words and phrases. weighted composition operator; numerical range; Hardy space.

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If T is a bounded linear operator on a Hilbert space H, the numerical range of T is the set $W(T) = \{\langle Tf, f\rangle : \|f\| = 1\}$. The set $W(T)$ is convex. There are some interesting papers where the numerical range of composition operators on H^2 was investigated (see [1], [2] and [6]). In this paper, we state some results of [3]. In Section 2, we work on the numerical range of $C_{\psi,\varphi}$, when φ is δ_r-conformal. Then, we show that $W(C_{\psi,wz})$ is a disk centered at 0, for $w \in \partial\mathbb{D}$, w not a root of unity, which is an improvement of [4, Theorem 3.17]. Next, we investigate the numerical range of weighted composition operator $C_{\psi,\varphi}$ that φ is an elliptic automorphism with rotation parameter w.

2. Main results

For $r \in \overline{\mathbb{D}}$, suppose that δ_r is the dilation which is defined by $\delta_r(z) = rz$. We say a map φ is δ_r-conformal if $\varphi = \alpha^{-1} \circ \delta_r \circ \alpha$, where α is an automorphism of $\mathbb{D}$ (in the case that $r \in \mathbb{D}$, Bourdon et al. in [2] called φ conformal dilation). We know that for some $p \in \mathbb{D}$ and $w \in \partial\mathbb{D}$, $\alpha = \rho_w \circ \alpha_p$, where ρ_w is the rotation which is defined by $\rho_w(z) = wz$ and $\alpha_p(z) = \frac{p-z}{1-\overline{p}z}$. Then it is not hard to see that $\varphi = \alpha_p \circ \rho_{\overline{w}} \circ \delta_r \circ \rho_{w\circ}\alpha_p = \alpha_p \circ \delta_r \circ \alpha_p$. It is easy to see that $\varphi'(p) = r$. Note that δ_r is a δ_r-dilation because $\delta_r = \alpha_0 \circ \delta_r \circ \alpha_0$. We call φ a positive conformal dilation when $0 < r < 1$. Bourdon et al. in [2, Theorem 4.4] showed that if φ is neither the identity map nor a positive conformal dilation and has a nonzero fixed point in $\mathbb{D}$, then 0 is an interior point of $W(C_\varphi)$. After that Gunatillake et al. in [4, Theorem 5.10] proved the similar result for weighted composition operators by the methods outlined in [2, Theorem 4.4], but in Theorem 2.1, we rewrite this result and also we show that [4, Theorem 5.10] is not correct for φ and ψ in the third part of Theorem 2.1. Moreover, if φ is the identity map and $\psi \in H^\infty$, then by [5, Corollary 2], $W(C_{\psi,\varphi}) = Hull(\psi(\mathbb{D}))$, but in Theorem 2.1 we do not assume that φ is the identity map.

Theorem 2.1. *Suppose that $C_{\psi,\varphi}$ is bounded and $\varphi(p) = p$, when $p \in \mathbb{D}$. Assume that $\widetilde{\psi} = c\frac{1-\overline{p}z}{1-\overline{p}rz} \circ \alpha_p$, where c is constant and $r \in \overline{\mathbb{D}}$. Then the following statements hold.*

(1) If φ is not δ_r-conformal with $-1 \le r \le 1$, then 0 is an interior point of $W(C_{\psi,\varphi})$.

(2) If $\psi \neq \widetilde{\psi}$ and φ is δ_r-conformal with $-1 \le r \le 0$, then 0 is an interior point of $W(C_{\psi,\varphi})$

(3) If $\psi = \widetilde{\psi}$ and φ is δ_r-conformal with $-1 \le r \le 0$, then $W(C_{\psi,\varphi})$ is a closed line segment with endpoints cr and c.

(4) If $\psi = \widetilde{\psi}$ *and* φ *is* δ_r*-conformal with* $0 < r < 1$*, then* $W(C_{\psi,\varphi})$ *is a half-open line segment with endpoints* 0 *and* c *that* $c \in W(C_{\psi,\varphi})$*.*

In [4, Theorem 3.17], the disks centered at 0 were found which contained in $W(C_{\psi,\varphi})$ for $\varphi(z) = wz$ and w not a root of unity. In the following proposition, we show that in this case $W(C_{\psi,\varphi})$ is a disk centered at 0.

Proposition 2.2. *Let* $\psi \in H^\infty$*. Suppose that* $|w| = 1$ *and* w *is not a root of unity. Then* $W(C_{\psi,wz})$ *is a disk centered at* 0*.*

Suppose that φ is an elliptic automorphism with the fixed point $p \in \mathbb{D}$. We know that φ must have the form $\varphi = \alpha_p \circ \delta_w \circ \alpha_p$, where $|w| = 1$, $\varphi'(p) = w$ and δ_w is the rotation. We call w the rotation parameter of φ. In [1, Theorem 4.1] Bourdon et al. showed that for an elliptic automorphism φ with a rotation parameter w which is not a root of unity, $\overline{W(C_\varphi)}$ is a disk centered at the origin. In the following theorem we state that this result holds for weighted composition operator. Furthermore, Gunatillake et al. in the third section of [4] computed $W(C_{\psi,\varphi})$ with rotational composition maps; Theorem 2.3 shows that those interesting results which were obtained in the third section of [4] will be useful in order to investigate $W(C_{\psi,\varphi})$, when φ is an elliptic automorphism.

Theorem 2.3. *Suppose that* φ *is an elliptic automorphism with the fixed point* $p \in \mathbb{D}$ *and the rotation parameter* w*. Assume that* $\psi \in H^\infty$*. Then* $W(C_{\psi,\varphi}) = W(T_{\psi\circ\alpha_p \cdot \frac{1-\bar{p}wz}{1-\bar{p}z}} C_{wz})$*. Moreover, if* w *is not a root of unity, then* $W(C_{\psi,\varphi})$ *is a disk centered at* 0*.*

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