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SOME ESSENTIALLY NORMAL WEIGHTED COMPOSITION OPERATORS ON THE WEIGHTED BERGMAN SPACES

MAHMOOD HAJI SHAABANI *

Department of Mathematics, Shiraz University of Technology, P. O. Box 71555-313, Shiraz, Iran.
shaabani@sutech.ac.ir

Abstract. First of all, we obtain a necessary and sufficient condition for a certain operator $T_w C_\varphi$ to be compact on A^2_α. Then, we characterize the essentially normal weighted composition operators $C_{\psi,\varphi}$ on the weighted Bergman spaces A^2_α, when $\varphi \in \mathrm{LFT}(\mathbb{D})$ is not an automorphism and $\psi \in H^\infty$ is continuous at a point ζ which φ has a finite angular derivative.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$. The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$.
Let dA be the normalized area measure in $\mathbb{D}$. The weighted Bergman spaces $A^2_\alpha(\mathbb{D}) = A^2_\alpha$, for $\alpha > -1$, are defined by

$$A^2_\alpha(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : ||f||^2_\alpha = \int_{\mathbb{D}} |f|^2 dA_\alpha < \infty\},$$

where $dA_\alpha = (\alpha+1)(1-|z|^2)^\alpha dA$. We know that for $\alpha > -1$, A^2_α is a Hilbert space (see, e.g., [3]).

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* Speaker.

Also we recall that $H^\infty(\mathbb{D}) = H^\infty$ is the space of all bounded analytic functions defined on $\mathbb{D}$, with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. Let φ be an analytic self-map of $\mathbb{D}$. We say that φ has an angular derivative at $\zeta \in \partial\mathbb{D}$ if there exists η on the unit circle such that the limit

$$\varphi'(\zeta) = \lim(\varphi(z) - \eta)/(z - \zeta),$$

where $z \to \zeta$ non-tangentially through $\mathbb{D}$, exists and is finite. When this happens it is clear that φ has non-tangential limit at ζ. Throughout this paper, let $F(\varphi)$ denote the set of all points in $\partial\mathbb{D}$ at which φ has a finite angular derivative.

Suppose that P_α is the projection of $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α. Given a function $w \in L^\infty(\mathbb{D})$, the Toeplitz operator T_w on A^2_α is defined by $T_w(f) = P_\alpha(wf)$. Since P_α is bounded on A^2_α, the Toeplitz operators are bounded.

For an analytic self-map φ of $\mathbb{D}$, we define the composition operator C_φ by $C_\varphi(f) = f \circ \varphi$, where f is an analytic map on $\mathbb{D}$. If ψ is in H^∞ and φ is an analytic map of the unit disk into itself, the weighted composition operators $C_{\psi,\varphi}$ is defined by $C_{\psi,\varphi}(f)(z) = \psi(z)f(\varphi(z))$, where f is analytic on $\mathbb{D}$. The weighted composition operators $C_{\psi,\varphi}$ are clearly bounded on A^2_α. Moreover, $C_{\psi,\varphi} = T_\psi C_\varphi$ on H^2 and A^2_α.

A linear-fractional self-map of $\mathbb{D}$ is a map of the form

$$\varphi(z) = \frac{az + b}{cz + d}, \tag{1.1}$$

for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, with the property that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. We denote the set of those maps by $\mathrm{LFT}(\mathbb{D})$. A map $\varphi \in \mathrm{LFT}(\mathbb{D})$ is called parabolic if it has a fixed point $\zeta \in \partial\mathbb{D}$ of multiplicity 2.

Recall that an operator A on a Hilbert space H is said to be normal if the self-commutator $AA^* - A^*A = 0$ on H and essentially normal if $AA^* - A^*A$ is compact on H. An operator A is nontrivially essentially normal if it is essentially normal, but neither normal nor compact. In [12], Nina Zorboska had shown that the essentially normal composition operators on the Hardy space H^2 with automorphism symbol are in fact normal. In addition, Zorboska had shown that the composition operators on H^2 induced by linear-fractional transformations fixing no point on the unit circle are not nontrivially essentially normal. Then, P. S. Bourdon, D. Levi, S. K. Narayan and J. H. Shapiro in [1] showed that when φ is a linear-fractional self-map of $\mathbb{D}$ with $\|\varphi\|_\infty = 1$, the operator C_φ is essentially normal on H^2 if and only if either φ is a rotation (in which case C_φ is normal), or φ is a parabolic non-automorphism self-map of the unit disk. In [9], B. D. MacCluer and R. J. Weir showed that the essentially normal linear-fractional composition operators on the weighted Bergman spaces A^2_α are exactly the same as those on the Hardy space; see also [8]. The essentially normal composition operators on other spaces have been investigated by some authors (see, e.g., [2], [5], [6] and [10]). In this paper, we state some results of [4].

2. Main results

In this section, we characterize the essentially normal weighted composition operators $C_{\psi,\varphi}$ on the weighted Bergman spaces A^2_α, whenever $\varphi \in \mathrm{LFT}(\mathbb{D})$ is not an automorphism and $\psi \in H^\infty$ is continuous at a point $\zeta \in F(\varphi)$.

Lemma 2.1. ([11, Corollary 1]) *Suppose $\alpha > -1$. Let φ and w be analytic functions on the unit disk, with w bounded and φ mapping the disk to itself. Then the weighted composition operator $T_w C_\varphi$ is compact on A^2_α if and only if*

$$\lim_{|z|\to 1} |w(z)|^2 \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

In the following proposition, we obtain a necessary and sufficient condition for a certain operator $T_w C_\varphi$ to be compact. Moreover, under the hypotheses of the following proposition, we show that the behavior of w at the points that are not in $F(\varphi)$ has no effect on the compactness of $T_w C_\varphi$ (see [7, Corollary 2.2] and [8, Theorem 3.3]). In the next proposition, the set of points which φ makes contact with $\partial\mathbb{D}$ is

$$\{\zeta \in \partial\mathbb{D} : \varphi(\zeta) \in \partial\mathbb{D}\}.$$

Proposition 2.2. *Let $\alpha > -1$ and φ be an analytic self-map of $\mathbb{D}$. Suppose that $\varphi \in A(\mathbb{D})$ and the set of points which φ makes contact with $\partial\mathbb{D}$ is finite. Let $w \in H^\infty$ be continuous at every point of $F(\varphi)$. Then $T_w C_\varphi$ is compact on A^2_α if and only if $w \equiv 0$ on $F(\varphi)$.*

In the next proposition, we show that the adjoint of C_φ, modulo the ideal of compact operators is a scalar multiple of C_σ.

Proposition 2.3. *Suppose that $\varphi(z) = \frac{az+b}{cz+d} \in LFT(\mathbb{D})$ is not an automorphism of $\mathbb{D}$ and that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. Let $\alpha > -1$ and $s = ((\overline{c}\overline{\zeta} + \overline{d})/(-\overline{b}\eta + \overline{d}))^{\alpha+2}$. Then there exists a compact operator K on A^2_α so that*

$$C^*_\varphi = sC_\sigma + K = |\varphi'(\zeta)|^{-(\alpha+2)} C_\sigma + K,$$

where $\sigma(z) = (\overline{a}z - \overline{c})/(-\overline{b}z + \overline{d})$.

Theorem 2.4. *Assume that* $\alpha > -1$. *Let* $\varphi \in LFT(\mathbb{D})$ *be a non-automorphism such that* $\varphi(\zeta) = \eta$ *for some* $\zeta, \eta \in \partial\mathbb{D}$. *Suppose that* $\psi \in H^\infty$ *is continuous at* ζ. *Then* $C_{\psi,\varphi}$ *is essentially normal on* A^2_α *if and only if* $\psi(\zeta) = 0$ *or* φ *is parabolic.*

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