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NORMS OF HYPONORMAL WEIGHTED COMPOSITION OPERATORS ON THE HARDY AND WEIGHTED BERGMAN SPACES

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Abstract. In this paper, first we find norms of hyponormal weighted composition operators $C_{\psi,\varphi}$, when φ has a Denjoy-Wolff point on the unit circle. Then for φ which is analytic self-map of $\mathbb{D}$ with a fixed point in $\mathbb{D}$, we investigate norms of hyponormal weighted composition operators $C_{\psi,\varphi}$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$. The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$. For f analytic on $\mathbb{D}$, we denote by $\hat{f}(n)$ the nth coefficient of the Maclaurin series of f. The Hardy space H^2 is the collection of all such functions f for which

$$\|f\|_1^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$

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For $\alpha > -1$, the weighted Bergman space A^2_α consists of all analytic f on $\mathbb{D}$ such that

$$\|f\|^2_{\alpha+2} = \int_{\mathbb{D}} |f(z)|^2(\alpha+1)(1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is normalized area measure on $\mathbb{D}$. Throughout this paper, let $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. We know that both the weighted Bergman space and the Hardy space are reproducing kernel Hilbert spaces, when the reproducing kernel for evaluation at w is given by $K_w(z) = (1 - \overline{w}z)^{-\gamma}$ for $z, w \in \mathbb{D}$. We write H^∞ for the space of bounded analytic functions on $\mathbb{D}$, with supremum norm $\|f\|_\infty$.

Let φ be an analytic self-map of $\mathbb{D}$. If H is a Hilbert space of analytic functions on $\mathbb{D}$, the composition operator C_φ on H is defined by the rule $C_\varphi(f) = f \circ \varphi$. Moreover, for an analytic function ψ on $\mathbb{D}$ and an analytic self-map φ of $\mathbb{D}$, we define the weighted composition operator $C_{\psi,\varphi}$ on H by $C_{\psi,\varphi} f = \psi(f \circ \varphi)$ for all $f \in H$. We say that φ has a finite angular derivative at $\zeta \in \partial\mathbb{D}$ if the nontangential limit $\varphi(\zeta)$ exists, has modulus 1, and $\varphi'(\zeta) = \angle \lim_{z\to\zeta} \frac{\varphi(z)-\varphi(\zeta)}{z-\zeta}$ exists and finite. Throughout this paper, let $F(\varphi)$ denote the set of all points in $\partial\mathbb{D}$ at which φ has a finite angular derivative. Let φ_0 be the identity map and φ_n denote the n-th iterate of φ. An elliptic automorphism is an automorphism linear-fractional which has one fixed point in the disk and the other is in the complement of the closed disk. It is well known that if φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map on the disk into itself, then there is a point c in $\overline{\mathbb{D}}$ so that the iterates φ_n of φ converge to c uniformly on compact subsets of $\overline{\mathbb{D}}$. The point c is called the Denjoy-Wolff point of φ. The Denjoy-Wolff point c is the unique fixed point of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(c)| \le 1$.

We say that an operator A on a Hilbert space H is hyponormal if $A^*A - AA^* \ge 0$, or equivalently if $\|A^*f\| \le \|Af\|$ for all $f \in H$. Cowen in [2, Theorem 5] provided a complete characterization of hyponormal composition operators C_φ on H^2 in the case where φ is linear-fractional (an analogue of [2, Theorem 5] for A^2_α has not been obtained yet). After that Zorboska [8] investigated the hyponormal composition operators on the weighted Hardy spaces. As far as we know, there is not a general characterization of hyponormal composition operators. Recently in [3], Cowen et al. investigated the situation where $C^*_{\psi,\varphi}$ is hyponormal. Moreover, hyponormal weighted composition have been investigated in some papers (see [6], [4]). In this paper, we state some results of [5].

2. Main results

Through this paper, the essential spectrum and the essential spectral radius of bounded operator T are denoted by $\sigma_{e,\gamma}(T)$ and $r_{e,\gamma}(T)$, respectively. Suppose that φ is an analytic self-map of $\mathbb{D}$ and α is a complex number of modulus 1. Since $\mathrm{Re}\left(\frac{\alpha+\varphi}{\alpha-\varphi}\right)$ is a positive harmonic function on $\mathbb{D}$, there exists a finite positive Borel measure μ_α on $\partial\mathbb{D}$ such that $\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \mathrm{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right) = \int_{\partial\mathbb{D}} P_z d\mu_\alpha$ for each $z \in \mathbb{D}$, where $P_z(e^{i\theta}) = (1-|z|^2)/|e^{i\theta}-z|^2$ is the Poisson kernel at z. The measures μ_α are called the Clark measures of φ. There is a unique pair of measures μ_α^{ac} and μ_α^s such that $\mu_\alpha = \mu_\alpha^{ac} + \mu_\alpha^s$, where μ_α^{ac} and μ_α^s are the absolutely continuous and singular parts with respect to Lebesgue measure, respectively. In particular, if φ is a linear-fractional non-automorphism such that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then $\mu_\alpha^s = 0$ when $\alpha \neq \eta$ and $\mu_\eta^s = |\varphi'(\zeta)|^{-1}\delta_\zeta$, where δ_ζ is the unit point mass at ζ. We write $E(\varphi)$ for the closure in $\partial\mathbb{D}$ of the union of the closed supports of μ_α^s as α ranges over the unit circle. We know that $F(\varphi) \subseteq E(\varphi)$ (see [7, p. 2919]). For information about the Clark measures, see [7].

In the next theorem, the set of points where the range of φ meets $\partial\mathbb{D}$ is

$$\{\zeta \in \partial\mathbb{D} : \varphi(\zeta) \in \partial\mathbb{D}\}.$$

Theorem 2.1. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that $\varphi \in A(\mathbb{D})$ and the set of points where the range of φ meets $\partial\mathbb{D}$ is finite. Assume that there are a positive integer n and $\zeta \in \partial\mathbb{D}$ such that $E(\varphi_n) = \{\zeta\}$, where ζ is the Denjoy-Wolff point of φ. Let $\psi \in H^\infty$ be continuous at ζ. If $C_{\psi,\varphi}$ is hyponormal, then*

$$\|C_{\psi,\varphi}\|_\gamma = |\psi(\zeta)|\varphi'(\zeta)^{-\gamma/2}.$$

In the next results, we consider hyponormal weighted composition operator $C_{\psi,\varphi}$, when φ has a Denjoy-Wolff point in $\mathbb{D}$.

Proposition 2.2. *Let φ be an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$ and $\psi \in H^\infty$. Suppose there is a positive integer n that $\{e^{i\theta} : |\varphi_n(e^{i\theta})| = 1\} = \emptyset$. If $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α, then $\|C_{\psi,\varphi}\|_\gamma = |\psi(0)|$.*

Proposition 2.3. *Let φ be analytic on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\varphi(0) = 0$. Assume that there is an integer n such that $\{e^{i\theta} : |\varphi_n(e^{i\theta})| = 1\}$ has only one element ζ which is a fixed point of φ and $\zeta \in F(\varphi)$. Suppose* *that $\varphi \in A(\mathbb{D})$ and $\psi \in H^\infty$ is continuous at ζ. If $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α, then*

$$\frac{|\psi(\zeta)|}{|\varphi'(\zeta)|^{\gamma/2}} \le \|C_{\psi,\varphi}\|_\gamma \le max\{|\psi(\zeta)|, |\psi(0)|\}.$$

Suppose that φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the unit disk into itself. In the following theorem, we see that if $C_{\psi,\varphi}$ is hyponormal, when $\varphi(p) = p$ for some $p \in \mathbb{D}$ and $r_{e,\gamma}(C_{\psi,\varphi}) \le |\psi(p)|$, then the function ψ has a simple linear-fractional form that is the same as what Bourdon et al. found in [1, Theorem 10]. Furthermore, in the next theorem, we find a necessary and sufficient condition for $C_{\psi,\varphi}$ to be hyponormal.

Theorem 2.4. *Suppose that φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the unit disk into itself with $\varphi(p) = p$, where $p \in \mathbb{D}$. Assume that $\psi \in H^\infty$. Suppose that for each $\lambda \in \sigma_{e,\gamma}(C_{\psi,\varphi})$, $|\lambda| \le |\psi(p)|$. The weighted composition operator $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α if and only if $\psi = \psi(p)\frac{K_p}{K_p\circ\varphi}$ and $C_{\alpha_p\circ\varphi\circ\alpha_p}$ is hyponormal, where $\alpha_p(z) = (p-z)/(1-\overline{p}z)$; moreover, in this case $\|C_{\psi,\varphi}\|_\gamma = |\psi(p)|$.*

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