Rglt-Majorization on $\mathbf{M}_{n,m}$ and its Linear Preservers

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Abstract. Let $\mathbf{M}_{n,m}$ be the set of all n-by-m real matrices. A matrix R is called generalized row stochastic (*g-row stochastic*) if the sum of entries on every row of R is one. For A, $B \in \mathbf{M}_{n,m}$, it is said that A is *rglt-majorized* by B, and we write $A \prec_{rglt} B$, if there exists an m-by-m lower triangular g-row stochastic matrix R so that $A = BR$. In this paper, the concept right lower triangular generalized row stochastic majorization, or rglt- majorization, is investigated and then the linear preservers and strong linear preservers of this concept are characterized on $\mathbb{R}_n$ and $\mathbf{M}_{n,m}$.

Keywords: G-row stochastic matrix, (Strong) linear preserver, Rglt-majorization.

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1. Introduction

Majorization is one of the vital topics in mathematics and statistics. It plays a basic role in matrix theory. For instance, majorization relation among eigenvalues and singular values of matrices produce a lot of norm inequalities. It was intensively studied in various directions; see, e.g., [1]-[12]. One of the directions is the study of linear functions that preserve or strongly preserve right matrix majorization ; see, e.g., [5] and [9].

Some of our notations and symbols are explained as the following: $\mathcal{R}_m^{gut}$ ($\mathcal{R}_m^{glt}$) for the collection of all m-by-m upper (lower) triangular g-row stochastic matrices; e for the column vector with all entries equal to one;E for the m-by-m matrix with all of the entries of the last column equal to one and the other entries equal to zero; E^* for the m-by-m matrix with all of the entries of the first column equal to one and the other entries equal to zero;$\mathbb{N}_k$ for the set $\{1, \ldots, k\} \subset \mathbb{N}$; $\mathcal{P}_n$ for the set of all n-by-n permutation matrices; $tr(x)$

for the summation of all components of a vector x in $\mathbb{R}_n$; P_n for the n-by-n backward identity matrix; $[T]$ for the matrix representation of a linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ with respect to the standard basis; $[T]_i$ for the i^{th} column of the matrix representation of a linear function T; r_i for the summation of all entries of i^{th} row of $[T]$; span(S) for the set of all linear combinations of the elements of S.

Let $\sim$ be a relation on $\mathbf{M}_{n,m}$. A linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ is said to be a linear preserver (or strong linear preserver) of $\sim$, if $TX \sim TY$ whenever $X \sim Y$ (or $TX \sim TY$ if and only if $X \sim Y$). Let $X, Y \in \mathbf{M}_{n,m}$. The matrix X is said to be *rgut-majorized* by Y (in symbol $X \prec_{rgut} Y$) if $X = YR$, for some $R \in \mathcal{R}_m^{gut}$.

2. Rglt-majorization

We intend to find all (strong) linear preservers of rglt-majorization on $\mathbb{R}_n$ and $\mathbf{M}_{n,m}$, too.

Definition 2.1. Let $x, y \in \mathbb{R}_n$. We say that x rglt-majorized by y (in symbol $x \prec_{rglt} y$) if $x = yR$, for some $R \in \mathcal{R}_n^{glt}$.

We bring the following propositions without proof.

Proposition 2.2. *Let A, $B \in \mathcal{R}_n^{glt}$. Then*
(a) $AB \in \mathcal{R}_n^{glt}$.
(b) If A is invertible, then $A^{-1} \in \mathcal{R}_n^{glt}$.

Assume that $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function. Define $\tau : \mathbb{R}_n \to \mathbb{R}_m$ by $\tau(x) = T(xP_n)P_m$.

Proposition 2.3. *Let x, $y \in \mathbb{R}_n$. Then $x \prec_{rgut} y$ if and only if $xP_n \prec_{rglt} yP_n$. Also, $xP_n \prec_{rgut} yP_n$ if and only if $x \prec_{rglt} y$.*

Proposition 2.4. *Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}_n$. Then $x \prec_{rglt} y$ if and only if $tr(x) = tr(y)$ and $x_i \in \operatorname{span}\{0, y_i, \dots, y_n\}$, for all $2 \leq i \leq n$.*

Now, we assert some prerequisites for introducing the main results of this section.

Proposition 2.5. *Let $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function. Then T preserves $\prec_{rgut}$ if and only if τ preserves $\prec_{rglt}$.*
Also, T preserves $\prec_{rglt}$ if and only if τ preserves $\prec_{rgut}$.

Define

$$\mathcal{A}_j(t_j) := \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ \alpha_1^j & \alpha_2^j & \dots & \alpha_{t_j}^j \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{pmatrix} \in \mathbf{M}_{n,t_j}, \tag{2.1}$$

where $j \geq 1$, $\alpha_{t_j}^j \neq 0$, $\begin{pmatrix} \alpha_1^j & \alpha_2^j & \dots & \alpha_{t_j}^j \end{pmatrix}$ is the $n-j+1^{th}$ row of $\mathcal{A}_j(t_j)$, and $*$ is a real number.

Also, define

$$\mathcal{B}_1(k_1) := \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{k_1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1 & \alpha_2 & \dots & \alpha_{k_1} \end{pmatrix} \in \mathbf{M}_{n,k_1}, \tag{2.2}$$

where $\alpha_{k_1} \neq 0$, and

$$\mathcal{B}_j(k_j) := \begin{pmatrix} \alpha_1^j & \alpha_2^j & \dots & \alpha_{k_j}^j \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^j & \alpha_2^j & \dots & \alpha_{k_j}^j \\ \beta_1^j & \beta_2^j & \dots & \beta_{k_j}^j \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{pmatrix} \in \mathbf{M}_{n,k_j}, \tag{2.3}$$

where $j \geq 2$, $\begin{pmatrix} \beta_1^j & \beta_2^j & \dots & \beta_{k_j}^j \end{pmatrix}$ is the $n-j+2^{th}$ row of $\mathcal{B}_j(k_j)$, and $\alpha_i^j \neq \beta_i^j$, for each $i \in \mathbb{N}_{k_j}$.

The following theorem characterizes structure of the linear functions $T : \mathbb{R}_n \to \mathbb{R}_m$ preserving rglt-majorization. The proofs are long. We have to give it up.

Theorem 2.6. *Let $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function. Then T preserves $\prec_{rglt}$ if and only if $r_1 = \dots = r_n$ and there exists a permutation matrix $P \in \mathcal{P}_m$ so that one of the following conditions occurs.*
a) $[T] = 0$,
b) $[T] = \big([T]_1 \quad * \quad \mathcal{B}_{n-1}(k_{n-1}) \quad \dots \quad \mathcal{B}_1(k_1)\big)\, P$,
c) $[T] = \big([T]_1 \quad \mathcal{B}_l(k_1) \quad \dots \quad \mathcal{B}_1(k_1)\big)\, P$,
d) $[T] = \big([T]_1 \quad * \quad \mathcal{A}_n(t_n) \quad \dots \quad \mathcal{A}_1(t_1)\big)\, P$,
where $\mathcal{B}_1(k_1)$, $\mathcal{B}_j(k_j)$ $(j \geq 2)$, and $\mathcal{A}_j(t_j)$ $(j \in \mathbb{N}_n)$ are the same as in (2.2),

(2.3), (2.1), respectively, in (b) $\sum_{j=1}^{n-1} k_j \leq m-1$ *and in* (c) $\sum_{j=1}^{l} k_j = m-1$.
e) $[T] = \begin{pmatrix} \mathcal{B} & 0 & \dots & 0 \\ * & \mathcal{A}_k(k_k)' & \dots & \mathcal{A}_1(t_1)' \end{pmatrix} P$, *where* $\mathcal{A}_j(t_j)' \in \boldsymbol{M}_{k,t_j}$ $(j \in \mathbb{N}_k)$ *is the same as in (2.1), and* $\mathcal{B} \in \boldsymbol{M}_{n-k,m-\sum_{j=1}^{k} t_j}$ *can be the zero matrix, or one of the forms* (b) *or* (c).

Theorem 2.7. *Let* $T : \mathbb{R}_n \to \mathbb{R}_n$ *be a linear function. Then* T *strongly preserves* $\prec_{rglt}$ *if and only if* $[T] = \alpha A$, *for some* $\alpha \in \mathbb{R} \setminus \{0\}$ *and an invertible matrix* $A \in \mathcal{R}_n^{glt}$.

The following theorem characterizes the strong linear preservers of $\prec_{rglt}$ on $\mathbf{M}_{n,m}$. We come to this without proof.

Theorem 2.8. *Let* $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ *be a linear function. Then* T *strongly preserves* $\prec_{rglt}$ *if and only if* $TX = RXA + SXE^*$, *for some* $R, S \in \boldsymbol{M}_n$ *and an invertible matrix* $A \in \mathcal{R}_m^{glt}$ *so that* $R(R+S)$ *is invertible.*

References

1. T. Ando, Majorization, Doubly stochastic matrices, and comparision of eigenvalues, *Linear Algebra and its Applications*, **118** (1989), 163-248.
2. A. Armandnejad and H. R. Afshin, Linear functions preserving multivariate and directional majorization, *Iranian Journal of Mathematical Sciences and Informatics*, **5**(1) (2010), 1-5.
3. A. Armandnejad and F. Pasandi, Block diagonal majorization, *Iranian Journal of Mathematical Sciences and Informatics*, **8**(2) (2013), 131-136.
4. A. Armandnejad and A. Ilkhanizadeh Manesh, Gut-majorization on $\mathbf{M}_{n,m}$ and its linear preservers, *Electronic Journal of Linear Algebra*, (23) (2012), 646-654.
5. A.M. Hasani and M. Radjabalipour, On linear preservers of (right) matrix majorization, *Linear Algebra and its Applications*, (423) (2007), 255-261.
6. A.M. Hasani and M. Radjabalipour, The structure of linear operators strongly preserving majorizations of matrices, *Electronic Journal of Linear Algebra* , (15) (2006), 260-268.
7. A. Ilkhanizadeh Manesh, On linear preservers of sgut-majorization on $\mathbf{M}_{n,m}$, *Bulletin of the Iranian Mathmatical Society,* **42** (2):470-481, 2016.
8. A. Ilkhanizadeh Manesh, Linear functions preserving sut-majorization on $\mathbb{R}^n$. *Iranian Journal of Mathematical Sciences and Informatics,* (to appear).
9. A. Ilkhanizadeh Manesh, Right gut-Majorization on $\mathbf{M}_{n,m}$, *Electronic Journal of Linear Algebra,* **31** (2016).
10. A. Ilkhanizadeh Manesh and A. Armandnejad, Ut-Majorization on $\mathbb{R}^n$ and its Linear Preservers, *Operator Theory: Advances and Applications,* (242) (2014), 253-259.
11. C.K. Li and E.Poon, Linear operators preserving directional majorization, *Linear Algebra and its Applications*, (235) (2001), 141-149.
12. A. W. Marshall, I. Olkin and B. C. Arnold, Inequalities: Theory of majorization and its applications, *Springer, New York*, (2011).