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# APPROXIMATE CONNES-BIPROJECTIVITY

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Abstract. In this paper, we introduce a notion of approximate Connes-biprojectivity for dual Banach algebras. We study the relation between approximate Connes-biprojectivity, approximate Connes amenability and $\varphi$-Connes amenability. We propose a criterion to show that certain dual triangular Banach algebras are not approximately Connes-biprojective. Next we show that for a locally compact group $G$, the Banach algebra $M(G)$ is approximately Connes-biprojective if and only if $G$ is amenable. Finally for an infinite commutative compact group $G$ we show that the Banach algebra $L^2(G)$ with convolution product is approximately Connes-biprojective, but it is not Connes-biprojective.

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## 1. Introduction

One of the most important notion in the theory of homological Banach algebras is biprojectivity which introduced by Helemskii [7]. Indeed, A Banach algebra $\mathcal{A}$ is called biprojective, if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$, such that $\rho$ is a right inverse for $\pi_{\mathcal{A}}$, where $\pi_{\mathcal{A}} : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is the product morphism which is given by $\pi_{\mathcal{A}}(a \otimes b) = ab$, for every $a, b \in \mathcal{A}$.

Recently, approximate homological notion like approximate biprojectivity of Banach algebras have been studied by Zhang [21]. Indeed, a Banach algebra $\mathcal{A}$ is called approximately biprojective, if there exists a net $(\rho_\alpha)$ of continuous $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho_\alpha(a) \to a$, for every $a \in \mathcal{A}$. For more information about approximate biprojectivity of some semigroup algebras, see [18].

There exists a class of Banach algebras which is called dual Banach algebras. This category of Banach algebras defined by Runde [14]. Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule $E$ is called dual if there is a closed submodule $E_*$ of $E^*$ such that $E = (E_*)^*$. The Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule. A dual Banach $\mathcal{A}$-bimodule $E$ is normal if for each $x \in E$ the module maps $\mathcal{A} \longrightarrow E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are $wk^*$-$wk^*$ continuous. Let $\mathcal{A}$ be a Banach algebra and let $E$ be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to E$ is called a bounded derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$, for every $a, b \in \mathcal{A}$. A bounded derivation $D : \mathcal{A} \to E$ is called inner if there exists an element $x$ in $E$ such that $D(a) = a \cdot x - x \cdot a$ $(a \in \mathcal{A})$. A dual Banach algebra $\mathcal{A}$ is called Connes amenable if for every normal dual Banach $\mathcal{A}$-bimodule $E$, every $wk^*$-continuous derivation $D : \mathcal{A} \longrightarrow E$ is inner. For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule $E$, $\sigma wc(E)$ denote the set of all elements $x \in E$ such that the module maps $\mathcal{A} \to E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are $wk^*$-$wk$-continuous. It is a closed submodule of $E$, see [14] and [15] for more details. Note that, since $\sigma wc(\mathcal{A}_*) = \mathcal{A}_*$, the adjoint of $\pi_{\mathcal{A}}$ maps $\mathcal{A}_*$ into $\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Therefore $\pi_{\mathcal{A}}^{**}$ drops to an $\mathcal{A}$-bimodule morphism $\pi_{\sigma wc} : (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \longrightarrow \mathcal{A}$. Every element $M \in (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ satisfying

$$a \cdot M = M \cdot a \quad \text{and} \quad a\pi_{\sigma wc}M = a \quad (a \in \mathcal{A}),$$

is called a $\sigma wc$-virtual diagonal for $\mathcal{A}$. Runde showed that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if there exists a $\sigma wc$-virtual diagonal for $\mathcal{A}$ [15, Theorem 4.8].

A dual Banach algebra $\mathcal{A}$ is called Connes-biprojective if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$ . Shirinkalam and the second author proved that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if $\mathcal{A}$ is Connes-biprojective and it has an identity [20]. They characterized Connes-biprojectivity of the measure algebra $M(G)$ for a locally compact group $G$. We extend this result for the following new notion.

Motivated by the definitions of approximate biprojectivity [21] and Connes-biprojectivity, we introduce a new class of dual Banach algebras.

**Definition 1.1.** A dual Banach algebra $\mathcal{A}$ is called approximately Connes-biprojective if there exists a (not necessarily bounded) net $(\rho_\alpha)_\alpha$ of continuous $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that

$$\pi_{\sigma wc}\circ\rho_\alpha(a)\to a \quad (a\in\mathcal{A}).$$

It is clear that every Connes-biprojective dual Banach algebra is approximately Connes-biprojective and the same result holds for every approximately biprojective dual Banach algebra.

In this paper we study the notion of approximately Connes-biprojectivity of dual Banach algebras. We show that there exists a relation between this new notion and $\varphi$-Connes amenability. Using this criterion, we investigate approximate Connes-biprojectivity of triangular Banach algebras. We study approximate Connes-biprojectivity of some dual Banach algebras associated with locally compact groups. More precisely, we show that for a locally compact group $G$, the measure algebra $M(G)$ is approximately Connes-biprojective if and only if $G$ is amenable. We extend the Example in [21, §2] to the approximately Connes-biprojective case and we show that for an infinite commutative compact group $G$ the Banach algebra $L^2(G)$ with convolution product is approximately Connes-biprojective, but not Connes-biprojective.

## 2. Approximate Connes-biprojectivity

For two Banach spaces $X$ and $Y$, we denote by $B(X,Y)$ the space of all bounded linear operators from $X$ to $Y$. Recall that the weak* operator topology ($W^*OT$) on $B(X,Y^*)$ is the locally convex topology determined by the seminorms $\{p_{x,y} : x\in X, y\in Y\}$, where $p_{x,y}(T)=|\langle y,Tx\rangle|$. Indeed the net $(T_\alpha)\subset B(X,Y^*)$ converges to $T$ in the weak* operator topology of $B(X,Y^*)$ if $T_\alpha(x)$ converges to $T(x)$ in the weak* topology of $Y^*$, for every $x\in X$.

*Remark* 2.1. If the net $(\rho_\alpha)$ in definition 1.1 is bounded, then the notions approximate Connes biprojectivity and Connes biprojectivity are the same. On bounded sets, the $W^*OT$ coincides with the $wk^*$-topology of $B(\mathcal{A},(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*)$, where identified with $(\mathcal{A}\hat{\otimes}\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$.

Since the unit ball of $B(\mathcal{A}, (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*)$ is $W^*OT$-compact, the net $(\rho_\alpha)$ has $W^*OT$-limit point. Let $\rho = W^*OT\text{-}\lim_\alpha \rho_\alpha$. It follows that

$$\rho(ab) = wk^*\text{-}\lim_\alpha \rho_\alpha(ab) = wk^*\text{-}\lim_\alpha a\cdot\rho_\alpha(b) = a\cdot wk^*\text{-}\lim_\alpha \rho_\alpha(b) = a\cdot\rho(b)$$

and by similarity for the right action, $\rho(ab) = \rho(a)\cdot b$. So $\rho : \mathcal{A} \to (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is a bounded $\mathcal{A}$-bimodule morphism and also

$$\pi_{\sigma wc} \circ \rho(a) = \pi_{\sigma wc}(wk^*\text{-}\lim_\alpha \rho_\alpha(a)) = wk^*\text{-}\lim_\alpha \pi_{\sigma wc} \circ \rho_\alpha(a) = a.$$

Some new generalization of Connes amenability like approximate Connes amenability have been introduced by Esslamzadeh *et al.* [6]. A unital dual Banach algebra $\mathcal{A}$ is approximately Connes amenable if and only if there exists a net $(M_\alpha)$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a\cdot M_\alpha - M_\alpha\cdot a \to 0$ and $\pi_{\sigma wc}(M_\alpha)a \to a$, for every $a \in \mathcal{A}$ [6, Theorem 3.3].

Note that a dual Banach algebra $\mathcal{A}$ has a bounded approximate identity if and only if $\mathcal{A}$ has an identity.

**Theorem 2.2.** *Let $\mathcal{A}$ be a dual Banach algebra. If $\mathcal{A}$ is approximately Connes-biprojective and has an identity, then $\mathcal{A}$ is approximately Connes amenable.*

We recall that a Banach algebra $\mathcal{A}$ is left $\varphi$-contractible, where $\varphi$ is a linear multiplication functional on $\mathcal{A}$, if there exists $m \in \mathcal{A}$ such that $am = \varphi(a)m$ and $\varphi(m) = 1$, for every $a \in \mathcal{A}$ [8], [11]. The notion of $\varphi$-Connes amenability for a dual Banach algebra $\mathcal{A}$, where $\varphi$ is a $wk^*$-continuous character on $\mathcal{A}$, was introduced by Mahmoodi and some characterizations were given [10]. We say that $\mathcal{A}$ is $\varphi$-Connes amenable if there exists a bounded linear functional $m$ on $\sigma wc(\mathcal{A}^*)$ satisfying $m(\varphi) = 1$ and $m(f\cdot a) = \varphi(a)m(f)$, for every $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$. It was shown by Ramezanpour [12, Proposition 2.3] and independently by Mahmoodi [9] that the concept of $\varphi$-Connes amenability is equivalent with left $\varphi$-contractibility for a dual Banach algebra, where $\varphi$ is a $wk^*$-continuous character . The set of all $wk^*$-continuous character on $\mathcal{A}$ is denoted by $\Delta_{wk^*}(\mathcal{A})$.

*Remark* 2.3. Let $\mathcal{A}$ be a dual Banach algebra and let $X$ be a Banach $\mathcal{A}$-bimodule. Since $\sigma wc(X^*)$ is a closed $\mathcal{A}$-submodule of $X^*$, we have a quotient map $q : X^{**} \longrightarrow (\sigma wc(X^*))^*$ defined by $q(u) = u|_{\sigma wc(X^*)}$, for every $u \in X^{**}$.

By inspiration of the main idea that used in [2, Proposition 2.8.41] and [17, Theorem 3.1], we state the following theorem.

**Theorem 2.4.** *Let $\mathcal{A}$ be an approximately Connes-biprojective dual Banach algebra and let $\varphi \in \Delta_{wk^*}(\mathcal{A})$ such that $\ker\varphi = \overline{\mathcal{A}\ker\varphi}$. Then $\mathcal{A}$ is left $\varphi$-contractible.*

**Example 2.5.** Consider the Banach algebra $\ell^1$ of all sequences $a = (a_n)$ of complex numbers with

$$\|a\| = \sum_{n=1}^{\infty} |a_n| < \infty,$$

and the following product

$$(a * b)(n) = \begin{cases} a(1)b(1) & \text{if} \quad n = 1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & \text{if} \quad n > 1 \end{cases}$$

for every $a, b \in \ell^1$. By simple argument $\ell^1$ is a dual Banach algebra with respect to $c_0$. We claim that $\ell^1$ is not approximately Connes-biprojective. We assume in contradiction that $\ell^1$ is approximately Connes-biprojective. Since $\ell^1$ is unital, by Theorem 2.4, $\ell^1$ is left $\varphi_1$-contractible, where $\varphi_1$ is a $wk^*$-continuous character on $\ell^1$ defined by $\varphi_1(a) = a(1)$. So there exists $m \in \ell^1$ satisfying

$$a * m = \varphi_1(a)m \quad \text{and} \quad \varphi_1(m) = m(1) = 1 \quad (a \in \ell^1). \tag{2.1}$$

Choose $a = \delta_n$, where $n \geq 2$. By (2.1), we have $\delta_n * m = 0$. It follows that $[m(1) + m(n)]\delta_n = 0$. Therefore $m(n) = -1$, for every $n \geq 2$, which is a contradiction with $\|m\|_1 < \infty$.

## 3. Applications to triangular Banach algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $X$ be a Banach $\mathcal{A} - \mathcal{B}$-module. That is, $X$ is a Banach left $\mathcal{A}$-module and a Banach right $\mathcal{B}$-module satisfying $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ and $||a \cdot x \cdot b|| \leq ||a|| \, ||x|| \, ||b||$, for every $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x \in X$. Consider

$$Tri(\mathcal{A}, \mathcal{B}, X) = \begin{pmatrix} \mathcal{A} & X \\ 0 & \mathcal{B} \end{pmatrix},$$

with the usual matrix operations and

$$|| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} || = ||a|| + ||x|| + ||b|| \quad (a \in \mathcal{A}, x \in X, b \in \mathcal{B}),$$

$Tri(\mathcal{A}, \mathcal{B}, X)$ becomes a Banach algebra which is called a triangular Banach algebra.

Note that if $\mathcal{A}$ is a dual Banach algebra, then $Tri(\mathcal{A}, \mathcal{A}, \mathcal{A})$ is a dual Banach algebra with respect to the predual $\mathcal{A}_* \oplus_\infty \mathcal{A}_* \oplus_\infty \mathcal{A}_*$.

**Theorem 3.1.** *Let $\mathcal{A}$ be a dual Banach algebra with a left approximate identity and let $\varphi \in \Delta_{wk^*}(\mathcal{A})$. Then $Tri(\mathcal{A}, \mathcal{A}, \mathcal{A})$ is not approximately Connes-biprojective.*

Let $S$ be the set of natural numbers $\mathbb{N}$ with maximum as its product. Then $S$ is a weakly cancellative semigroup, that is, for every $s, t \in S$ the set $\{x \in S : sx = t\}$ is finite [3, Example 3.36]. $\ell^1(S)$ is a dual Banach algebra with predual $c_0(S)$ [3, Theorem 4.6].

**Corollary 3.2.** *Let $S = (\mathbb{N}, \max)$. Then $Tri(\ell^1(S), \ell^1(S), \ell^1(S))$ is not approximately Connes-biprojective.*

Let $\mathcal{A}$ be a Banach algebra and let $E$ be a Banach $\mathcal{A}$-bimodule. An element $x \in E$ is called weakly almost periodic if the module maps $\mathcal{A} \longrightarrow E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are weakly compact. The set of all weakly almost periodic elements of $E$ is denoted by $WAP(E)$ which is a norm closed sub-bimodule of $E$ [15, Definition 4.1]. For a Banach algebra $\mathcal{A}$, Runde observed that $F(\mathcal{A}) = WAP(\mathcal{A}^*)^*$ is a dual Banach algebra with the first Arens product inherited from $\mathcal{A}^{**}$. He also showed that $F(\mathcal{A})$ is a canonical dual Banach algebra associated to $\mathcal{A}$ [15, Theorem 4.10].

We recall that if $E$ is a Banach $\mathcal{A}$-bimodule, then $E^*$ is also a Banach $\mathcal{A}$-bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \qquad (f \cdot a)(x) = f(a \cdot x) \quad (a \in \mathcal{A}, x \in E, f \in E^*).$$

**Corollary 3.3.** *Let $S = (\mathbb{N}, \min)$. Then $Tri(F(\ell^1(S)), F(\ell^1(S)), F(\ell^1(S)))$ is not approximately Connes-biprojective.*

## 4. Applications to some Banach algebras related to a locally compact group

Let $G$ be a locally compact group. It is well-known that the measure algebra $M(G)$ is a dual Banach algebra [16, Example 4.4.2].

**Theorem 4.1.** *For a locally compact group $G$, the followings are equivalent:*

(i) *$G$ is amenable,*
(ii) *$M(G)$ is approximately Connes-biprojective.*

Zhang showed that the Banach algebra $\ell^2(S)$ with the pointwise multiplication is approximately biprojective but it is not biprojective [21, §2]. We extend this example to the approximately Connes-biprojective case.

Note that $\ell^p(S)$ for $1 \leq p < \infty$ and arbitrary set $S$ with pointwise multiplication is a dual Banach algebra.

**Theorem 4.2.** *Let $S$ be an infinite set. Then $\ell^2(S)$ is approximately Connes-biprojective but it is not Connes-biprojective.*

Let $G$ be a locally compact group. Rickert showed that $L^2(G)$ is a Banach algebra with convolution if and only if $G$ is compact [13].

The proof of the following Corollary is similar to the [19, Theorem 2.17]. Therefore we omit it:

**Corollary 4.3.** *Let $G$ be an infinite commutative compact group. Then $L^2(G)$ with convolution is approximately Connes-biprojective, but it is not Connes-biprojective.*

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