The Extended Abstracts of
The 6$^{\text{th}}$ Seminar on Functional Analysis and its Applications
4-5th March 2020, University of Isfahan, Iran

WOVEN FRAMES AND THEIRS DUALITY

ELAHE AGHESHTE MOGHADAM *, [1] ALI AKBAR AREFEEJAMAL [2]

[1] *Department of Mathematics, University of Hakim Sabzevari*
elahe.moghadam@gmail
[2] *Department of Mathematics, University of Hakim Sabzevari*
arefijamaal@hsu.ac.ir

Abstract. Weaving frames in separable Hilbert spaces have been recently introduced by Bemrose et al. to deal with some problems in distributed signal processing and wireless sensor networks. In this paper, we obtain necessary and sufficient conditions under which a frame and its (canonical) dual are woven.

1. Introduction

Frames which are introduced by Duffin and Schaefer provide robust, stable and usually non-unique representations of vectors [4]. So far, the theory of frames has been growing rapidly. They have seen great achievements in pure mathematics, science, and engineering such as image processing, signal processing, sampling and approximation theory [3]. For example in signal processing each signal is interpreted as a vector. In this interpretation, a vector expressed as a linear combination of the frame vectors. Using a frame, it is possible to create a simpler, more sparse representation of a signal as compared with a family of elementary signals.

2010 *Mathematics Subject Classification.* 42C15.
Key words and phrases. Frames; Dual of frames; Woven frames; Dual of woven frames .
* Speaker.

A sequence $\Phi = \{\varphi_i\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants $0 < A_\Phi \leq B_\Phi < \infty$ such that

$$A_\Phi \|f\|^2 \leq \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \leq B_\Phi \|f\|^2, \qquad (f \in \mathcal{H}). \tag{1.1}$$

The constants A_Φ and B_Φ are called upper and lower frame bounds, respectively. A sequence $\{\varphi_i\}_{i\in I}$ is called *Bessel* if the right inequality in (1.1) holds.

A sequence $\Phi = \{\varphi_i\}_{i\in I}$ in Hilbert space $\mathcal{H}$ is called a *Riesz sequence* if there are constants $0 < A_\Phi \leq B_\Phi < \infty$ so that for all finite scalars c_i we have

$$A_\Phi \textstyle\sum_{i\in I} |c_i|^2 \leq \| \sum_{i\in I} c_i\varphi_i \|^2 \leq B_\Phi \sum_{i\in I} |c_i|^2$$

where A_Φ and B_Φ are *lower* and *upper Riesz bounds*, respectively. In addition, if Φ is complete in $\mathcal{H}$, then it is a *Riesz basis* for $\mathcal{H}$.

Given a Bessel sequence $\Phi = \{\varphi_i\}_{i\in I}$, the *synthesis operator* $T_\Phi : \ell^2(\mathbb{N}) \to \mathcal{H}$ is defined by $T_\Phi\{c_i\} = \sum_{i\in I} c_i\varphi_i$. Its adjoint, $T_\Phi^* : \mathcal{H} \to \ell^2(\mathbb{N})$, which is called the *analysis operator*, is given by $T_\Phi^* f = \{\langle f, \varphi_i\rangle\}$. Moreover, the *frame operator* of Φ, $S_\Phi : \mathcal{H} \to \mathcal{H}$, is given by $S_\Phi f = T_\Phi T_\Phi^* f$. If Φ is a frame, then S_Φ is invertible and $A_\Phi \leq S_\Phi \leq B_\Phi$, see [5] for more details.

The sequence $\tilde{\Phi} = \{S_\Phi^{-1}\varphi_i\}_{i\in I}$, which is also a frame, is called the *canonical dual frame*. A frame $\{\psi_i\}_{i\in I}$ is called a *dual* of $\{\varphi_i\}_{i\in I}$ if

$$f = \textstyle\sum_{i\in I} \langle f, \psi_i\rangle \varphi_i, \qquad (f \in \mathcal{H}).$$

Theorem 1.1. [2] *If $\Phi = \{\varphi_i\}_{i\in I}$ be a frame. Then every dual frame of Φ is of the form of $\Phi^d = \{S_\Phi^{-1}\varphi_i + u_i\}_{i\in I}$, where $\{u_i\}_{i\in I}$ is a Bessel sequence such that $\sum_{i\in I} \langle f, u_i\rangle \varphi_i = 0$.*

The concept of *woven frames* is motivated by the following question in signal processing: given two frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$. At each sensor we measure a signal f either with φ_i or with ψ_i, so that the collected information is the set of numbers $\{< f, \varphi_i >\}_{i\in\sigma} \cup \{< f, \psi_i >\}_{i\in\sigma^c}$ for some subset $\sigma \subset I$. Can f still be recovered from these measurements, no matter which kind of measurement has been made at each sensor? In other words, is the set $\{\varphi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ a frame for all subset $\sigma \subset I$? This question led us to the definition of woven frames.

Two frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ for Hilbert space $\mathcal{H}$ are *weakly woven* if for every subset $\sigma \subset I$, the family $\{\varphi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for $\mathcal{H}$.

Proposition 1.2. *Let $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be frames for Hilbert space $\mathcal{H}$ such that there is a $0 < \lambda < 1$ so that*

$$\lambda(\sqrt{B_\Phi} + \sqrt{B_\Psi}) \leq \frac{A_\Phi}{2}$$

and for all sequence of scalars $\{a_i\}_{i\in I}$ we have

$$\| \sum_{i\in I} a_i(\varphi_i - \psi_i) \| \leq \lambda \| \{a_i\}_{i\in I} \|$$

Then for every $\sigma \subset I$, the family $\{\varphi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for $\mathcal{H}$ with frame bounds $\frac{A_\Phi}{2}$ and $B_\Phi + B_\Psi$. That is, Φ and Ψ are woven.

Proposition 1.3. *Let $\Phi = \{\varphi_i\}$ be a frame and U be a bounded operator such that $\|I_{\mathcal{H}} - U\|^2 < \frac{A_\Phi}{B_\Phi}$. Then Φ and $U\phi$ are woven with the universal lower bound $\left(\sqrt{A_\Phi} - \sqrt{B_\Phi}\|I_{\mathcal{H}} - U^*\|\right)^2$.*

Two frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are called *Riesz woven* if for every subset $\sigma \subset I$, the family $\{\varphi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a Riesz basis for $\mathcal{H}$. Combining Theorem 5.3 of [3] and Theorem 2.5 of [1], every Riesz basis with its canonical dual is Riesz woven. However, the canonical Parseval dual of two Riesz woven bases are not necessary woven. For example, $\{e_1, e_2\}$ and $\{e_1+e_2, 2e_1+e_2\}$ are Riesz woven bases with the canonical Parseval $\{e_1, e_2\}$ and $\{e_2, e_1\}$ which are clearly not woven [3]. One of our aim is to find dual frames which are woven.

2. Main results

The concept of weaving not only depends on the properties of two frames, but also depends on the order of elements. For example, two orthonormal bases are not necessarily woven. Thus, dual frames and the images of a frame under bounded operators are the best candidates for weaving by a given frame.

In the next example we show that the woven property is not transitive, in general.

Example 2.1. Let $\mathcal{H} = \mathbb{R}^2$. Consider frames $\Phi = \{e_1, e_1, e_2\}$, $\Psi = \{e_1, e_2, e_2\}$ and $\eta = \{e_1, e_2, e_1\}$ on $\mathcal{H}$ where $\{e_1, e_2\}$ is the standard orthonormal basis of $\mathcal{H}$. Then Φ with Ψ and Ψ with η are woven with universal bounds $A_1 = A_2 = 1$ and $B_1 = B_2 = 2$. However, Φ and η are not woven.

In the following theorem, we show that under some conditions the woven property is an equivalence relation.

Theorem 2.2. *Let $\{\varphi_i\}_{i\in I}$, $\{\psi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ be frames for Hilbert space $\mathcal{H}$. If $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven frames with the universal bounds A_1, B_1, and $\{\psi_i\}_{i\in I}$ with $\{\eta_i\}_{i\in I}$ are woven with universal bounds A_2 and B_2 such that $A_1+A_2-B_\psi>0$, then $\{\varphi_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ are woven.*

Corollary 2.3. *Let $\Phi=\{\varphi_i\}_{i\in I}$ and $\Psi=\{\psi_i\}_{i\in I}$ be woven with universal bounds A_1 and B_1 and $\Psi^d=\{\psi_i^d\}_{i\in I}$ be a dual of Ψ. If Ψ and Ψ^d be woven with universal bounds A_2 and B_2 such that $A_1+A_2-B_\Psi>0$. then Φ and Ψ^d are woven.*

In the next theorem, we show that if two frames Φ and Ψ are woven then there is a dual Φ^d such that ψ and $S_\Phi\Phi^d$ are woven.

Theorem 2.4. *let $\Phi=\{\varphi_i\}_{i\in I}$ and $\Psi=\{\psi_i\}_{i\in I}$ be a pair of woven frames for $\mathcal{H}$. Then there is a dual frame of Φ as Φ^d for which Ψ and $S_\Phi\Phi^d$ are woven.*

In Proposition 1.3, it is shown that under some condition a frame and its canonical dual frame are woven. In the next theorem, we obtain this result by a sharper bound.

Theorem 2.5. *Let $\Phi=\{\varphi_i\}_{i\in I}$ be a frame such that*

$$\| I-S_\Phi^{-1} \| < \frac{A_\Phi}{2(B_\Phi+\sqrt{\frac{B_\Phi}{A_\Phi}})}. \tag{2.1}$$

Then Φ and $\tilde{\Phi}$ are woven.

In [1], it is shown that a frame with the small norm of redundant elements is woven by its canonical dual. In the following we prove this fact for alternate duals.

Theorem 2.6. *Let $\Phi=\{\varphi_i\}_{i\in I}$ be a frame for $\mathcal{H}$. If the norm of redundant elements of Φ are small enough, then there is a dual frame Φ^d such that Φ and Φ^d are woven.*

References

1. Arabyani Neyshaburi, F. and Arefijamaal, A., Manufactoring Pairs of Woven Frames Applying Duality Principle on Hilbert Spaces, submitted.
2. Arefijamaal, A. and Zekaee, E., Signal processing by alternate dual Gabor frames, Appl. Comp. Harmon. Anal. 35, 535-540, 2013.
3. Bemrose, T., Casazza, P. G., Grochenig, K., Lammers, M. C. and Lynch, R. G., Weaving Hilbert space frames, Operators and Matrices. 10 (4), 1093-1116, 2016.
4. Duffin, R. and Schaeffer, A., A class of nonharmonic Fourier serier series, Trans. Amer. Math. soc. 72, 341-366, 1952.
5. Christensen, O., Frames and Bases: An Inteoductory course, Birkhäuser, Boston.2008.