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COMPOSITION OPERATORS ON VARIABLE EXPONENT BERGMAN SPACES

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Abstract. In this paper we study composition operators on variable exponent Bergman spaces in the unit disk of the complex plane. We prove that this operators are bounded, and we give a sufficient condition for the compactness of composition operators.

1. Introduction

Let $\Omega \subseteq R^n$. We mean by a *variable exponent*, a measurable function $p : \Omega \to [1, \infty)$. We shall write

$$p_+ = p_\Omega^+ := ess\ sup_{x\in\Omega}(p(x)),$$
$$p_- = p_\Omega^- := ess\ inf_{x\in\Omega}(p(x)).$$

Let $\mathcal{P}(\Omega)$ denote the set of all variable exponent functions p(.) for which $p_+ < \infty$. For a complex-valued measurable function $f : \Omega \to \mathbb{C}$ we define the modular $\rho_{p(.)}$ by

$$\rho_{p(.),\mu}(f) := \int_\Omega |f(x)|^{p(x)} d\mu(x),$$

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where μ is the Lebesgue measure on Ω. The *Luxemburg-Nakano* norm induced by this modular is given by

$$\|f\|_{L^{p(.)}} := \inf_{\lambda} \left\{ \lambda > 0 : \rho_{p(.),\mu} \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

From now on we let $\Omega = \mathbb{D}$ be the open unit disk equipped with the normalized area measure $d\mu(z) = dA(z) = \pi^{-1} dx\, dy$. We denote the modular by $\rho_{p(.)}$ and the induced norm by $\|.\|_{p(.)}$. It is well-known that (see [1]) the dual of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. We also have

$$\begin{aligned} (p'(.))_+ &= \left(p'_-\right), \\ (p'(.))_- &= \left(p'_+\right). \end{aligned}$$

For abbreviation we shall write $p'_+(.)$ and $p'_-(.)$.

Definition 1.1. Let $p(.) \in \mathcal{P}(\Omega)$. $L^{p(\cdot)}(\Omega, \mu)$ is the space consisting of all complex-valued measurable functions $f : \Omega \to \mathbb{C}$ that satisfy $\rho_{p(.),\mu}(f) < \infty$. We know from [1] that $L^{p(\cdot)}(\Omega, \mu)$ is Banach space.

Definition 1.2. A function $p : \Omega \to \mathbb{R}$ is said to be *locally log-holder continuous* on Ω if there exist a positive constant C such that for $x, y \in \Omega$ with $|x - y| < \frac{1}{2}$ we have

$$|p(x) - p(y)| \leq \frac{C}{\log(|x - y|)}.$$

We denote by $P^{\log}(\Omega)$ the set of all locally log-holder continuous functions in Ω for which $1 < p_- \leq p_+ < \infty$.

Definition 1.3. Given $p(.) \in P(\mathbb{D})$ we define the *variable exponent Bergman space* $A^{p(\cdot)}(\mathbb{D})$ as the space of all analytic functions on $\mathbb{D}$ for which

$$\int_D |f(z)|^{p(z)} dA(z) < \infty.$$

With this definition, $A^{p(\cdot)}(\mathbb{D})$ is a closed subspace of $L^{p(\cdot)}(\mathbb{D})$ and hence a Banach space.

2. Main results

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an holomorphic function on $\mathbb{D}$. We consider the induced composition operator $C_\varphi : A^{p(\cdot)}(\mathbb{D}) \to A^{p(\cdot)}(\mathbb{D})$, that is:

$$(C_\varphi f)(z) = (f \circ \varphi)(z) = f(\varphi(z)), \qquad f \in A^{p(\cdot)}(\mathbb{D}), \quad z \in \mathbb{D}.$$

In the first theorem, we prove that the operator C_φ is bounded on $A^{p(\cdot)}(\mathbb{D})$. This will generalize the result proved in [5] for constant exponent $1 < p < \infty$.

Theorem 2.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function and $p(.) \in \mathcal{P}^{\log}(\mathbb{D})$ then composition operator $C_\varphi : A^{p(\cdot)}(\mathbb{D}) \to A^{p(\cdot)}(\mathbb{D})$ is bounded.*

Now we turn to the issue of compactness. To begin, we need to recall some items from [2] and [4].

Let φ_z be a Mobius function on $\mathbb{D}$, that is, $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$. We then define the operator U_z on $A^{p(\cdot)}$ by

$$U_z f := (f \circ \varphi_z)\varphi_z', \qquad z \in \mathbb{D}.$$

In [2] A. Dieudonne proved that U_z is a bounded on $A^{p(\cdot)}$.

Let S be a bounded operator on $A^{p(\cdot)}$, we define operator

$$S_{(z)} := U_z S U_z.$$

Let k_z be normalized Bergman kernel on the unit disk; that is:

$$k_z(w) = \left(1 - |z|^2\right) \frac{1}{(1 - \bar{z}w)^2}.$$

We now define function $\tilde{S}$ on $\mathbb{D}$:

$$\tilde{S}(z) := \langle S k_z, k_z \rangle.$$

In fact this function is well known Berezin transform of S on $A^2(\mathbb{D})$. The next theorem is a basic tool in the following arguments; indeed, it is an $A^{p(\cdot)}$ version of a result already proved by Miao and Zhang [4] for the constant variable Bergman space $A^p(\mathbb{D})$.

Theorem 2.2. ([2]) Suppose $p(.) \in \mathcal{P}^{\log}(\mathbb{D})$, $1 < p_0 \leq p_- \leq p_+ < \infty$, and $p_1 = \min\{p_0, p_0'\}$. Let S be a bounded operator on $A^{p(\cdot)}$ such that for some $q > \frac{p_1+1}{p_1-1}$,

$$C_1 = \sup_{z\in\mathbb{D}} \|S_{(z)}1\|_q < \infty,$$
$$C_2 = \sup_{z\in\mathbb{D}} \left\|S^*_{(z)}1\right\|_q < \infty \left(S^*_{(z)} \text{is the adjoint of } S_{(z)}\right).$$

Then the following statements are equivalent:

(1) S is compact on $A^{p(\cdot)}$.
(2) $\tilde{S}(z) \to 0$ as $z \to \partial\mathbb{D}$.
(3) For every $s \in [1, q)$, $\|S_{(z)}1\|_s \to 0$ as $z \to \partial\mathbb{D}$.
(4) $\|S_{(z)}1\|_1 \to 0$ as $z \to \partial\mathbb{D}$.

To prove our main result, we need the following lemma.

Lemma 2.3. *Suppose* $p(.) \in \mathcal{P}^{\log}(\mathbb{D})$, $1 < p_0 \leq p_- \leq p_+ < \infty$, *and* $p_1 = \min\{p_0, p_0'\}$. *Suppose there exist* $q > \frac{p_1+1}{p_1-1}$ *such that*

$$\lim_{|z|\to 1^-} \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{(1-|z\varphi(\omega)|)^2 q} dA(\omega) = 0.$$

Then we have $\sup_{z\in\mathbb{D}} \|C_{\varphi_{(z)}} 1\|_q < \infty$ *and* $\sup_{z\in\mathbb{D}} \|C^*_{\varphi_{(z)}} 1\|_q < \infty$.

Theorem 2.4. *Suppose* $p(.) \in \mathcal{P}^{\log}(\mathbb{D})$, $1 < p_0 \leq p_- \leq p_+ < \infty$, *and* $p_1 = \min\{p_0, p_0'\}$. *Let* $\varphi : D \to D$ *is a holomorphic function in the unit disk for which there exists* $q > \frac{p_1+1}{p_1-1}$ *satisfying*

$$\lim_{|z|\to 1^-} \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{1-|\varphi(\omega)|)^{2q}} dA(\omega) = 0.$$

Then the composition operator $C_\varphi : A^{p(.)}(D) \to A^{p(.)}(D)$ *is compact on* $A^{p(.)}$.

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