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ON MODULI OF p-VARIATION AND CONVERGENCE OF FOURIER SERIES

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Abstract. In this talk, by using the notion of l_p-modulus of variation of a function ($p \geq 1$), we introduce a Banach space of functions of generalized bounded variation, denoted $V_p[\nu]$, and consider several problems pertinent to the theory of Fourier series in this space. In particular, we first state a Helly-type result and an embedding theorem for $V_p[\nu]$. Next, under some restrictions on p and ν, we give a characterization of the uniform convergence of Fourier series in $H^\omega \cap V_p[\nu]$, where H^ω is the class of all functions whose moduli of continuity (in the classical sense) are dominated by ω. In conclusion, an estimate on the magnitude of the Fourier coefficients in $V_p[\nu]$ is obtained.

1. Introduction

The theory of Fourier series began with the significant work of Fourier on heat conduction at the beginning of the 19th century. Although his idea of expanding a periodic function in terms of trigonometric series

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fails in general, it has been found to work in various classes of functions. For instance, the Dirichlet–Jordan Theorem states that functions of bounded variation (BV) have convergent Fourier series.

Meanwhile, the classical notion of bounded variation, due to Jordan, has been generalized in many ways and studied extensively in regard to the theory of Fourier series. An early step in extending Jordan's variation was the introduction of bounded p-variation by Wiener. Afterwards, Young [5], Waterman [4], Schramm [3] and others considered various such generalizations of Jordan's variation. On the other hand, Chanturiya [2] considered modulus of variation of a function and utilized it in establishing an estimate on the "distance" between a function and partial sums of its Fourier series. (See [1] for more information.)

In this talk, by using the notion of l_p-modulus of variation of a function, we introduce a class of functions of generalized bounded variation, denoted $V_p[\nu]$, and consider several problems pertinent to the theory of Fourier series in this class.

2. Preliminaries

This section is devoted to some preliminary notions and definitions that are necessary in formulating our results.

Definition 2.1. A sequence ν of positive real numbers is called a modulus of variation if it is nondecreasing and concave. The l_p-modulus of variation of a function f on $[a, b]$ is the sequence

$$v_p(n, f) := \sup \Big(\sum_{j=1}^{n} |f(I_j)|^p \Big)^{\frac{1}{p}},$$

where the supremum is taken over all finite collections $\{I_j\}_{j=1}^n$ of nonoverlapping subintervals of $[a, b]$ and $f(I_j) = f(\sup I_j) - f(\inf I_j)$.

The symbol $V_p[\nu]$ denotes the class of all functions f such that $\nu_p(n, f) = O(\nu(n))$ as $n \to \infty$. When $p = 1$ we get the Chanturiya class $V[\nu]$, and when $\nu(n) = 1$ we have the Wiener class V_p.

Remark 2.2. One can observe that $V_p[\nu]$ is a linear space, and the norm

$$\|f\|_{p,\nu} := |f(a)| + \sup_{1 \le n < \infty} \frac{v_p(n, f)}{\nu(n)} < \infty,$$

makes $V_p[\nu]$ into a Banach space.

Definition 2.3. Let $\Phi = \{\phi_j\}_{j=1}^{\infty}$ be a sequence of increasing convex functions on the nonnegative real numbers such that $\phi_j(0) = 0$ for all j. We say that Φ is a Φ-sequence if $0 < \phi_{j+1}(x) \le \phi_j(x)$ for all j and

$\sum_{j=1}^{\infty} \phi_j(x) = \infty$ for all $x > 0$. A real function f on $[a, b]$ is said to be of Φ-bounded variation if

$$\mathrm{Var}_{\Phi}(f) := \sup \sum_{j=1}^{n} \phi_j(|f(I_j)|) < \infty,$$

where the supremum is taken over all finite collections $\{I_j\}_{j=1}^{\infty}$ of nonoverlapping subintervals of $[a, b]$. In the sequel, we assume that $[a, b] = [0, 2\pi]$ and that all functions f are 2π-periodic.

We denote by ΦBV the linear space of all functions f such that cf is of Φ-bounded variation for some $c > 0$, which was originally introduced and studied in [3]. Indeed, ΦBV turns into a Banach space—with a suitable norm—in which the Helly's selection theorem holds.

Definition 2.4. The modulus of continuity of a function f is defined as follows:

$$\omega(\delta, f) := \sup_{0 \le h \le \delta} \sup_{x \in [0, 2\pi]} |f(x + h) - f(x)|, \quad \delta \ge 0.$$

By a modulus of continuity we mean a continuous, subadditive and nondecreasing function ω on the nonnegative real numbers such that $\omega(0) = 0$. The symbol H^{ω} stands for the class of all 2π-periodic functions for which $\omega(\delta, f) = O(\omega(\delta))$ as $\delta \to 0^{+}$.

3. Main results

It is well-known that Helly's selection theorem holds for functions of bounded variation. Our first result states that an analogue of that theorem is valid for $V_p[\nu]$.

Theorem 3.1. *Let $\{f_n\}$ be a bounded sequence in $V_p[\nu]$ with $\|f_n\|_{p,\nu} \le M$. Then there exists a subsequence $\{f_{n_k}\}$ converging pointwise to a function f in $V_p[\nu]$ such that $\|f\|_{p,\nu} \le 2M$.*

Next theorem provides a necessary and sufficient condition for the embedding $\Phi\mathrm{BV} \hookrightarrow V_p[\nu]$. For each k, $\Phi_k^{-1}(x)$ denotes the inverse of the function $\Phi_k(x) := \sum_{j=1}^{k} \phi_j(x)$, $x \ge 0$.

Theorem 3.2. *Let Φ be a Φ-sequence, ν be a modulus of variation and $1 \le p < \infty$. Then ΦBV embeds into $V_p[\nu]$ if and only if*

$$\limsup_{n \to \infty} \frac{1}{\nu(n)} \max_{1 \le k \le n} k^{\frac{1}{p}} \Phi_k^{-1}(1) < \infty.$$

Let ω be a modulus of continuity, ν be a modulus of variation, and $1 \le p < \infty$. In the following theorem we give a necessary and sufficient condition for uniform convergence of the Fourier series of functions in the class $H^\omega \cap V_p[\nu]$, under the assumptions that $\nu(n) \sim n^\alpha$ for some $0 < \alpha < \frac{1}{p}$.

Notation: In order to formulate the main result of this talk, we introduce a handy piece of notation:

$$\sigma(n) := \min_{1 \leq r \leq n-1} \left\{ \omega\left(\frac{1}{n}\right) \sum_{k=1}^{r} \frac{1}{k} + \sum_{k=r+1}^{n-1} \frac{\nu(k)}{k^{1+\frac{1}{p}}} \right\},$$

and

$$\tau(n) := \omega\left(\frac{1}{n}\right) \sum_{k=1}^{\theta} \frac{1}{k} + \sum_{k=\theta+1}^{n-1} \frac{\nu(k) - \nu(k-1)}{k^{\frac{1}{p}}},$$

where the minimum in the definition of $\sigma(n)$ is attained at $r = \theta$.

Theorem 3.3. *Let ω, ν, p and α be as above. Then the following are equivalent.*

(a) *The Fourier series of every function in the class $H^\omega \cap V_p[\nu]$ converges uniformly.*

(b) $\lim_{n\to\infty} \sigma(n) = 0$.

(c) $\lim_{n\to\infty} \tau(n) = 0$.

In conclusion, an estimate is given on the order of magnitude of Fourier coefficients in $V_p[\nu]$.

Proposition 3.4. *Let ν be a modulus of variation and $1 \leq p < \infty$. If $f \in V_p[\nu]$ is 2π-periodic, then*

$$|\hat{f}(n)| = O\Big(\frac{\nu(n)}{n^{\frac{1}{p}}}\Big) \quad \textit{as } n \to \infty.$$

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