The Extended Abstracts of
The 6$^{\text{th}}$ Seminar on Functional Analysis and its Applications
4-5th March 2020, University of Isfahan, Iran

SURVEY ON COMPLETENESS OF TRACE CLASS FOR PROPER H^*-ALGEBRA

AKBAR NAZARI[1], MOHAMMAD AMIN MOARREFI[*2]

[1] *Department of Pure Mathematices, Faculty of Mathematics & Computer, Shahid Bahonar University of Kerman, Kerman, Iran.*
nazari@uk.ac.ir
[2] *Department of Pure Mathematices, Faculty of Mathematics & Computer, Shahid Bahonar University of Kerman, Kerman, Iran.*
ma_moarrefi@yahoo.com

Abstract. H^*-algebra is defined by Warren Ambrose. These structure are complete algebra w.r.t. norm $\|\cdot\|_{\mathcal{A}}$ that corresponding with inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$. Trace class is a sub-algebra of H^*-algebra and define on it a trace norm $\tau(\cdot)$. The space $(\tau(\mathcal{A}),\tau(\cdot))$ is a complete sub-algebra of H^*-algebra.

1. Introduction

An algebra is a vector space $\mathcal{A}$ with a multiplication $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ s.t. $(a,b)\mapsto ab$ which is associative and linear in each of the two variables of multiplication operator. A Banach algebra is an algebra $\mathcal{A}$ over field $\mathbb{F}$ with equipped the norm $\|\cdot\|_{\mathcal{A}}$ that is a Banach space such that for all $a,b\in\mathcal{A}$, $\|ab\|_{\mathcal{A}}\leq\|a\|_{\mathcal{A}}\|b\|_{\mathcal{A}}$. An unital Banach algebra is a Banach algebra with a unit element $1_{\mathcal{A}}$ such that $\|1_{\mathcal{A}}\|_{\mathcal{A}}=1_{\mathbb{F}}$. An involution is a map $a\mapsto a^*$ from $\mathcal{A}$ into $\mathcal{A}$ such that $(a^*)^*=a$, $(ab)^*=b^*a^*$ and $(\alpha a+b)^*=\bar{\alpha}a^*+b^*$, for $a,b\in\mathcal{A}$ and $\alpha\in\mathbb{C}$. Each Banach

2010 *Mathematics Subject Classification.* Primary: 06F25; Secondary: 16R30.
Key words and phrases. algebra; H^*-algebra; trace class.
* Speaker.

algebra equipped with an involution is called Banach $*$-algebra or B^*-algebra. A C^*-algebra is a Banach algebra $\mathcal{A}$ with an involution such that $\|a^*a\|_{\mathcal{A}} = \|a\|^2_{\mathcal{A}}$ for every $a \in \mathcal{A}$.

2. H^*-Algebra

H^*-algebra is defined by Warren Ambrose in [1]. We survey some properties of this algebra. Also we checking the difference between that and the similar algebra in an example.

Definition 2.1. Banach algebra $\mathcal{A}$ is called H^*-algebra if:

i. The underlying Banach space of $\mathcal{A}$ is Hilbert space.
ii. For each $a \in \mathcal{A}$, there exist adjoint $a^* \in \mathcal{A}$ such that
$$\langle ab, c\rangle_{\mathcal{A}} = \langle b, a^*c\rangle_{\mathcal{A}} \quad \text{and} \quad \langle ab, c\rangle_{\mathcal{A}} = \langle a, cb^*\rangle_{\mathcal{A}} \tag{2.1}$$
for all $a, b, c \in \mathcal{A}$.

This means, the algebra norm $\|a\|_{\mathcal{A}}$ and the Hilbert space norm $\langle a, a\rangle^{1/2}_{\mathcal{A}}$ are equal. Also, the adjoint a^* of a may not be unique.

Example 2.2. We explain two example for H^*-algebra and relation between this and C^*-algebra:

i. Complex number, $\mathbb{C}$, with inner product $\langle \alpha, \beta\rangle = \mathrm{Re}\,(\alpha\beta^*)$ and induced norm by this, is an H^*-algebra and C^*-algebra.
ii. The Clifford algebra $\mathcal{A} = Cl_{0,n}$ is a real H^*-algebra w.r.t. $\langle \lambda, \mu\rangle := 2^n[\lambda\overline{\mu}]_0 = 2^n \sum_A \lambda_A \mu_A$. Then $|\lambda|^2_0 := \langle \lambda, \lambda\rangle_0 = 2^n \sum_A \lambda^2_A$ be an induced norm by above inner product on $C\ell_{02}$ (i.e. $i^2 = j^2 = -1$). Assume $\lambda = i + j$, then $\lambda\overline{\lambda} = 2$, $|\lambda\overline{\lambda}|_0 = 4$ and $|\lambda|^2_0 = 8$. Hence $|\lambda\overline{\lambda}|_0 \neq |\lambda|^2_0$. Therefore $Cl_{0,n}$ with this norm is not a C^*-algebra (For more information see [2]).

Let $\mathcal{A}$ be an H^*-algebra and $a \in \mathcal{A}$. Then $a\mathcal{A} = \{0\}$ is equivalent to $\mathcal{A}a = \{0\}$. Define $Z := \{a \in \mathcal{A} | a\mathcal{A} = \{0\}\}$.

Definition 2.3. An H^*-algebra is proper or semi-simple if $Z = \{0\}$.

Theorem 2.4. *An H^*-algebra is proper if and only if every element has a unique adjoint.*

Definition 2.5. Let $\mathcal{A}$ be an H^*-algebra and $a, e, f \in \mathcal{A}$. Then a is self-adjoint member of $\mathcal{A}$ if $a^* = a$. a is positive member of $\mathcal{A}$ if $\langle ax, x\rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{A}$. a is normal element if $a^*a = aa^*$. e is idempotent if $e^2 = e \neq 0$. e is sa-idempotent (projection) if e be an idempotent and a self-adjoint element. Idempotents e, f are called doubly orthogonal if $ef = fe = 0$ and $\langle e, f\rangle_{\mathcal{A}} = 0$. An idempotent is primitive if it can not be expressed as the sum of two doubly orthogonal idempotents.

Theorem 2.6. *Every proper H^*-algebra contains a non-empty maximal family of doubly orthogonal primitive sa-idempotents.*

Definition 2.7. Let T is a bounded linear operator in Banach algebra X, then T will be called a right centralizer of X if T satisfies the identity $T(xy) = (Tx)y$. We will use the symbol $R(X)$ to denote the collection of all right centralizers of X.

Let a be arbitrary and $La : \mathcal{A} \to \mathcal{A}$ be the operator of the left multiplication by a, i.e. $La(x) := ax$. Then $La \in R(\mathcal{A})$. We define $C(\mathcal{A})$ to be subspace generated by the operators La, $a \in \mathcal{A}$. $C(\mathcal{A})$ is the closed subspace of $R(\mathcal{A})$ in the operator norm.

2.1. **Trace-Class for H^*-Algebras.** In continuous, $\mathcal{A}$ is a proper H^*-algebra. We discuass about trace–class and trace–functional.

Saworotnow show that for each $a \neq 0$ in $\mathcal{A}$ there exists a sequence $\{e_n\}$ of mutually orthogonal projections and a sequence $\{\lambda_n\}$ of positive numbers such that $a^*a = \sum_n \lambda_n e_n$. Also that $a^*ae_n = e_n a^*a = \lambda_n e_n$ for each n. Then they define $[a] := \sum_n \mu_n e_n$, where $\mu_n := \sqrt{\lambda_n} \geq 0$. For each $a \in \mathcal{A}$ there exists a unique positive member $[a]$ of $\mathcal{A}$ such that $[a]^2 = a^*a$ (note that $[a]^* = [a]$).

Definition 2.8. Trace-class for $\mathcal{A}$ is the set $\tau(\mathcal{A}) = \{xy | x, y \in \mathcal{A}\}$. Also, if $a = xy \in \tau(\mathcal{A})$, define $\operatorname{tr} a := \langle y, x^* \rangle_{\mathcal{A}}$.

Trace tr is a positive functional, i.e. if $a \in \mathcal{A}$ be a positive then $\operatorname{tr}(a) \geq 0$. There exists $b \in \mathcal{A}$ such that $\operatorname{tr}(b) < 0$. Therefore in follow use "[$\cdot$]" to build a norm of this functional. For every $a \in \mathcal{A}$, $[a]$ is a positive member of $\mathcal{A}$.

Definition 2.9. With above assumption, we define $\tau(a) := \operatorname{tr}([a]) = \operatorname{tr}(\sum_{n=1}^{\infty} \mu_n e_n) = \sum_{n=1}^{\infty} \mu_n$ for every $a \in \mathcal{A}$.

Corollary 2.10. *Suppose $\mathcal{A}$ be an H^*-algebra. Then*

i. $\tau(a^*a) = \operatorname{tr}(a^*a) = \|a\|_{\mathcal{A}}^2$, *for all* $a \in \mathcal{A}$;
ii. $|tra| \leq \tau(a)$, *for all* $a \in \tau(\mathcal{A})$;
iii. *If* $a \in \tau(\mathcal{A})$ *and* S *is a right centralizer then* $\tau(Sa) \leq \|S\| \, \tau(a)$;
iv. $\|a\|_{\mathcal{A}} \leq \tau(a)$, *for all* $a \in \tau(\mathcal{A})$;
v. $\tau(ab) \leq \|a\|_{\mathcal{A}} \cdot \|b\|_{\mathcal{A}}$, *for all* $a, b \in \mathcal{A}$;
vi. $\tau(ab) \leq \tau(a)\tau(b)$, *for all* $a, b \in \tau(\mathcal{A})$.

3. Completeness of Trace Class

In this section, investigated trace-class space $\tau(\mathcal{A})$ with norm $\tau(\cdot)$ and show that $(\tau(\mathcal{A}), \tau(\cdot))$ is a complete space. Therefore by Proposition 2.10, part (vi), this space is Banach algebra.

Lemma 3.1. *If $a \in \tau(\mathcal{A})$ then the mapping f_a defined on $C(\mathcal{A})$ with $f_a(S) = \operatorname{tr}(Sa)$ is a bounded linear functional and $\|f_a\| = \tau(a)$.*

Theorem 3.2. *Each bounded linear functional on $C(\mathcal{A})$ is of the form f_a for some $a \in \tau(\mathcal{A})$.*

This means, the above correspondence between $\tau(\mathcal{A})$ and $C(\mathcal{A})^*$ is an isometric isomorphism. $\tau(\mathcal{A})$ can be identified with the space of all bounded linear functionals on $C(\mathcal{A})$.

Corollary 3.3. *$\tau(\mathcal{A})$ is a Banach algebra in the norm $\tau(\cdot)$.*

Proof. $\tau(\mathcal{A})$ is complete since it is isometric to the dual of $C(\mathcal{A})$. □

Theorem 3.4. *For every right centeralizer S the mapping $f_S(x) = \operatorname{tr}(Sx)$, is a bounded linear functional on $\tau(\mathcal{A})$ such that $\|f_S\| = \|S\|$. Conversely, each bounded linear functional on $\tau(\mathcal{A})$ is of the form f_S for some $S \in R(\mathcal{A})$. Thus $R(\mathcal{A})$ is isometric isomorphic to $\tau(\mathcal{A})^*$.*

Example 3.5. Consider standard structure $\ell_2(\mathbb{N})$ with the common addition and scalar product. Suppose $a = (a_1, a_2, \cdots) = \{a_i\}_{i=1}^{\infty}, b = (b_1, b_2, \cdots) = \{b_i\}_{i=1}^{\infty} \in \ell_2(\mathbb{N})$ where $a_i, b_i \in \mathbb{F} = \mathbb{C}$. We define $a \cdot b := \{a_i \cdot b_i\}_{i=1}^{\infty}$, $\langle a, b\rangle_{\mathcal{A}} := \sum_{i=1}^{\infty} a_i\overline{b_i}$ and $a^* := (\overline{a_1}, \overline{a_2}, \cdots)$ where $\overline{a_i}$ is conjugate of complex number a_i. Then $\|a\|_{\mathcal{A}} = \langle a, a\rangle_{\mathcal{A}}^{1/2} = (\sum_{n=1}^{\infty} |a_n|^2)^{1/2}$ is induced norm. $\mathcal{A}$ is an H^*-algebra, because $\langle ab, c\rangle_{\mathcal{A}} = \langle b, a^*c\rangle_{\mathcal{A}} = \langle a, cb^*\rangle_{\mathcal{A}} = \sum a_i b_i \bar{c}_i$, but it has not C^*-algebra structure. For check this, let $a = (2, 3, 0, 0, \cdots)$. Then $\|a\|_{\mathcal{A}}^2 \neq \|a^*a\|_{\mathcal{A}}$.

Let $\delta_i = \{\delta_{ij}\}_{j=1}^{\infty}$, $i \in \mathbb{N}$, where δ_{ij} is the Kronecker delta. Then $\{\delta_i\}$ is family of doubly orthogonal primitive sa-idempotents.

For every $a = \{a_n\}_{n=1}^{\infty}$, $[a] = \{|a_n|\}_{n=1}^{\infty}$. Then $\operatorname{tr}(a) = \sum_{n=1}^{\infty} a_n$ and $\tau(a) = \sum_{n=1}^{\infty} |a_n|$. Therefore $\|a\|_{\mathcal{A}} = \langle a, a\rangle_{\mathcal{A}}^{1/2} = \sqrt{\tau(a^*a)}$.

Finally, $(\tau(\mathcal{A}), \tau(\cdot))$ is a Banach algebra, but is not a C^*-algebra.

References

1. W. Ambrose, Structure theorems for a special class of Banach algebras, *Trans. Amer. Math. Soc.*, 57 (1945), 364–386.
2. M. A. Moarrefi and A. Nazari, Investigate of the C^*-algebraic Structure of the Clifford Algebra, 49th Annual Iranian Mathematics Conference, Elm-o-Sanat University, Tehran, Iran, pp 3069–3075, 2018.
3. L. Molnar, Modular bases in a hilbert A-module, *Czechoslovak Mathematical Journal*, 42 (1992), 649–656.
4. A. Nazari and M. A. Moarrefi, Survey on trace class for proper H^*-algebra, 10th Seminar on Linar Algebra and its Applications, Shahid Bahonar University, Kerman, Iran, 2020 (Submitted).
5. P. P. Saworotnow, Trace-class and centralizers of an H^*-algebra, *Proc. Amer. Math. Soc.*, 26 (1970), 101–104.