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GENERALIZED NONEXPANSIVE MAPPINGS AND FIXED POINTS

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Abstract. In this talk, we study the relationship between some generalization of nonexpansive mappings and obtain some fixed point theorems for these class of mappings. As a corollary we show that every Hilbert space and every l_p space, where $1 < p \leq 2$, have the fixed point property for (α, β)-nonexpansive mapping with $\alpha > 0$ and $\beta \geq 0$ such that $\alpha + \beta < 1$.

1. Introduction

Let C be a nonempty subset of a Banach space $(X, \|.\|)$ and let $T : C \to X$. Recall that T is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for each $x, y \in C$. A point $p \in C$ is called a fixed point for T when $Tp = p$ and the fixed point set of T is denoted by $\mathrm{Fix}(T)$. Also T is said to be a quasi-nonexpansive, if $\mathrm{Fix}(T) \neq \emptyset$ and for every $p \in \mathrm{Fix}(T)$ and $x \in C$, $\|Tx - p\| \leq \|x - p\|$.

Fixed point theorems for nonexpansive mappings and generalization of nonexpansive mappings is an important topic in metric fixed point theory. Therefore in the recent years, several generalizations of nonexpansive mappings have received attention and their fixed point theory

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and the relationship between them, have been studied by many authors, see [1, 2, 3] and the references therein.

In the following we have some definitions and results which will be used in the sequel.

Definition 1.1. [5] We will say that a Banach space X satisfies the *Opial's condition* if for every weakly null sequence (x_n) and every $x \neq 0$ in X, we have

$$\liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x_n + x\|.$$

Every Hilbert space satisfies the Opial's condition [4].
Let C be a nonempty subset of Banach space X and $T : C \to C$ be a mapping. The sequence $(x_n) \subseteq C$ is called an *approximate fixed point sequence* (a.f.p.s) for T provided that $\|x_n - Tx_n\| \to 0$.

Definition 1.2. [3] Let C be a nonempty subset of Banach space X. We say that the mapping $T : C \to X$ satisfies condition (A) on C whenever there exists an a.f.p.s for T in each nonempty bounded closed convex and T-invariant subset D of C ($Tx \subseteq D$ for any $x \in D$).

Definition 1.3. [3] Let C be a nonempty bounded closed convex subset of Banach space X. A mapping $T : C \to X$ satisfies condition (L),(or it is an (L)-type mapping), on C provided that it fulfills condition (A) on C and for any a.f.p.s (x_n) of T in C and each $x \in C$,

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

Definition 1.4. Let X be a Banach space and $\mathcal{F}$ be a class of mapping on X. We say that X has *fixed point property* (FPP) for $\mathcal{F}$, if for every nonempty weakly compact convex subset C of X every mapping $T : C \to C$ of $\mathcal{F}$ has a fixed point.

2. Main results

In 2018, Amini-Harandi et. al. introduced a two parametric class of nonlinear mappings [1].

Definition 2.1. [1] Let C be a nonempty subset of a Banach space X and let $\alpha, \beta \in \mathbb{R}$. A mapping $T : C \to X$ is said to be (α, β)-*nonexpansive mapping* if, for each $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \alpha\|y - Tx\|^2 + \alpha\|x - Ty\|^2 + \beta\|x - Tx\|^2 + \beta\|y - Ty\|^2 + (1 - 2\alpha - 2\beta)\|x - y\|^2.$$

Now we obtain a characterization of (α, β)-nonexpansive mapping in Hilbert spaces.

Definition 2.2. Let H be a Hilbert space and let C be a nonempty subset of H. Let λ and μ be two real number. A mapping $T : C \to H$ is said to be (λ, μ)-*hybrid* if

$$\|Tx-Ty\|^2 \leq \|x-y\|^2+2(1-\lambda)\langle x-Tx, y-Ty\rangle+2(1-\mu)\langle x-Ty, y-Tx\rangle,$$

for each $x, y \in C$.

Proposition 2.3. *Let H be a Hilbert space, C be a nonempty subset of H and let $T : C \to H$ be a mapping. Let λ and μ be two real numbers such that $\lambda + \mu < 3$ and put $\alpha = \frac{1-\lambda}{3-\lambda-\mu}$ and $\beta = \frac{1-\mu}{3-\lambda-\mu}$. Then T is (λ, μ)-hybrid if and only if T be an (α, β)-nonexpansive mapping.*

In Example 2.1 of [1] we see that, there exists a $(\frac{1}{1000}, \frac{8}{9})$-nonexpansive mapping, which is not an (L)-type mapping. Now we introduce a new class of generalized nonexpansive mapping, which is an extention of the (L)-type mapping and contain the the class of (α, β)-nonexpansive mapping with $\alpha > 0$ and $\beta \geq 0$ such that $\alpha + \beta < 1$.

Definition 2.4. Let C be a nonempty subset of Banach space X. A mapping $T : C \to X$ is called (H)-*type mapping* if it fulfills condition (A) on C and there exists $0 \leq \lambda < 1$, such that for any a.f.p.s (x_n) of T in C and each $x \in C$,

$$\limsup_{n\to\infty} \|x_n - Tx\|^2 \leq \limsup_{n\to\infty} \|x_n - x\|^2 + \lambda\|x - Tx\|^2.$$

Proposition 2.5. *Let C be a nonempty bounded subset of Banach space X and let $T : C \to X$ be an (α, β)-nonexpansive mapping with $\alpha > 0$ and $\beta \geq 0$ such that $\alpha + \beta < 1$. Then T is an (H)-type mapping.*

Remark 2.6. In Example 2.1 of [1] there exists a $(\frac{1}{1000}, \frac{8}{9})$-nonexpansive mapping, which is not quasi-nonexpansive. Then by Proposition 2.5, this mapping is an (H)-type mapping which is not (L)-type nor quasi-nonexpansive.

Theorem 2.7. *Every Hilbert space and every l_p space, where $1 < p \leq 2$, have FPP for (H)-type mapping.*

Corollary 2.8. *Every Hilbert space and every l_p space, where $1 < p \leq 2$, have FPP for (α, β)-nonexpansive mapping with $\alpha > 0$ and $\beta \geq 0$ such that $\alpha + \beta < 1$.*

Corollary 2.9. *Let C be a nonempty weakly compact convex subset of Hilbert space H or l_p space, where $1 < p \leq 2$ and let $T : C \to C$ be an (H)-type mapping. Then $I - T$ is demiclosed, that is, if (x_n) be a.f.p.s of T that is weakly convergent to $x \in C$, then $x \in FixT$.*

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