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A VERSION OF FARAKAS'S LEMMA IN VECTOR SPACES

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Abstract. In this paper, we consider the generalized vector equilibrium problems in the setting of vector spaces. Then we establish a set-valued version of Farakas's lemma in the setting of real ordered vector spaces. By using it, we give several optimality conditions for solutions of generalized vector equilibrium problems.
Keywords: Algebraic interior, Generalized vector equilibrium problems, Pointed convex cones, Farkas's Lemma.

1. Introduction

A large number of research papers have been published on different aspects of vector equilibrium problems, see, for example [1] and the references therein. Some of the generalizations of vector equilibrium problems are listed below.

$$\text{Find } x \in K \text{ such that } F(x,y) \subseteq Y \backslash (-\operatorname{int} C), \quad \forall y \in K, \tag{1.1}$$

$$\text{Find } x \in K \text{ such that } F(x,y) \nsubseteq -\operatorname{int} C, \quad \forall y \in K, \tag{1.2}$$

$$\text{Find } x \in K \text{ such that } F(x,y) \cap Y \backslash (-\operatorname{int} C) \neq \emptyset \quad \forall y \in K, \tag{1.3}$$

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$$\text{Find } x \in K \text{ such that } F(x,y) \subseteq C, \quad \forall y \in K, \tag{1.4}$$

where K is nonempty set, $F : K \times K \rightrightarrows Y$ is a set-valued mapping with nonempty values, and C is a convex cone in a topological vector space Y with nonempty interior, denoted by $\operatorname{int} C$. These problems are called generalized vector equilibrium problems (in short, GVEP). Clearly the solution set of (1.1) is a subset of (1.2) and the solution set of (1.2) is a subset of (1.3). It is worth to mention that GVEP are the tools to study vector optimization problems for nondifferentiable and nonconvex vector-valued functions. Section 2 contains notations, definitions and preliminary results which will be used in the sequel. In Section 3, we first establish a set-valued version of Farakas lemma in the setting of ordered vector spaces. Then by using it, we give several optimality conditions for a solution of generalized vector equilibrium problems in the setting of vector spaces.

2. Preliminaries and Formulations

All vector spaces in this paper are real. Let X, Y and Z be ordered vector spaces and $P \subseteq Y$ and $Q \subseteq Z$ be pointed convex cones. We denote by Y^* and Z^* the algebraic dual spaces of Y and Z, respectively. If A is a nonempty subset of Y, then the generated cone of A is defined as $\operatorname{cone}A = \bigcup_{\lambda \geq 0} \lambda A = \{\lambda a : \lambda \geq 0,\, a \in A\}$. The algebraic dual cone P^* and strictly dual cone $P^{\#}$ of P are defined as

$$P^* = \{y^* \in Y^* : \langle y^*, p\rangle \geq 0 \text{ for all } p \in P\},$$

and

$$P^{\#} = \{y^* \in Y^* : \langle y^*, p\rangle > 0, \forall p \in P \setminus \{0\}\},$$

where $\langle y^*, p\rangle$ denotes the value of the linear functional y^* at the point p, and 0 denotes the zero vector of the corresponding vector space.

The algebraic interior of A, denoted by $\operatorname{cor}A$, is defined as

$$\operatorname{cor}A = \{a \in A : \forall y \in Y, \exists\, \delta_0 > 0, \forall \delta \in [0, \delta_0], a + \delta y \in A\}.$$

Let Y be a topological vector space and A be a nonempty subset of Y. Then the topological interior of A, denoted by $\operatorname{int}A$, is a subset of $\operatorname{cor}A$.

Lemma 2.1. [2] *Let A be a nonempty convex subset of a topological vector space X such that* $\operatorname{int} A \neq \emptyset$. *Then the following assertions hold.*

(a) $intA = corA$.
(b) $clA = cl(intA)$ *and* $intA = int(clA)$, *where* clA *denotes the closure of the set* A.

Definition 2.2. (GWVEP) A vector $x \in K$ satisfying

$$F(x, y) \nsubseteq -\operatorname{cor} P$$

for all $y \in K$, is called a weakly efficient solution to the VEP.

Definition 2.3. (GVEP) A vector $x \in K$ is called a globally efficient solution to the VEP if there exists a pointed convex cone $H \subset Y$ with $P \setminus \{0\} \subset corH$ such that

$$F(x, K) \cap ((-H) \setminus \{0\}) = \emptyset$$

where $F(x, K) = \bigcup_{y \in K} F(x, y)$.

Definition 2.4. (HGVEP) A vector $x \in K$ is called a Henig efficient solution to the VEP if there exists an algebraic open set U containing 0 with $U \subset V_B$ satisfying

$$coneF(x, K) \cap (-corP_U(B)) = \emptyset.$$

Definition 2.5. (SGVEP) A vector $x \in K$ is called a suppereefficient solution to the VEP if for eath algebraic open set V of 0, there exists an algebraic open set U of 0 satisfying

$$coneF(x, K) \cap (U - P) \subset V$$

Clearly,

$$coneF(x, K) \cap (-corP_U(B)) \subset coneF(x, K) \cap (U - P).$$

3. Optimality Conditions

Let X be a vector space, Y and Z be ordered vector spaces, $P \subseteq Y$ and $Q \subseteq Z$ be pointed convex cones, $K \subseteq X$ be a nonempty set, and $F : K \rightrightarrows Y$ and $G : K \rightrightarrows Z$ be set-valued mappings with nonempty values. Define

$$\langle F(x), y^* \rangle = \{\langle y, y^* \rangle \,:\, y \in F(x)\} \quad \text{and} \quad \langle F(K), y^* \rangle = \bigcup_{x \in K} \langle F(x), y^* \rangle .$$

We write

$$\begin{aligned} F(x) <_P y_0 \quad &\text{if and only if} \quad y <_P y_0, \quad \forall y \in F(x), \\ F(x) \leq_P y_0 \quad &\text{if and only if} \quad y \leq_P y_0, \quad \forall y \in F(x). \end{aligned}$$

Theorem 3.1. *(Generalized Farkas-Minkowski Theorem) Let K be a nonempty convex subset of X. If $F : K \rightrightarrows Y$ is $P-$convex, $G : K \rightrightarrows Z$ is $Q-$convex, and the system*

$$\begin{cases} F(x) <_P 0, \\ G(x) <_Q 0, \end{cases}$$

has no solution in K, then there exists nonzero element $(y^*, z^*) \in P^* \times Q^*$ such that for all $x \in K$,

$$\langle y^*, F(x)\rangle + \langle z^*, G(x)\rangle \geq 0,$$
$$\textit{that is, } \langle y^*, y\rangle + \langle z^*, z\rangle \geq 0, \quad \textit{for all } y \in F(x),\ z \in G(x),$$

where $F(x) <_P 0$ and $G(x) <_Q 0$ mean that $F(x) \subset -corP$ and $G(x) \subset -corQ$, respectively.

Let $F : K \times K \rightrightarrows Y$ be a set-valued mapping and P is a convex cone in Y. Assume that $G : K \rightrightarrows Z$ is a set-valued mapping with nonempty values and Q is a convex cone in Z. We introduce the following condition which is needed in the sequel.

(Assumption C) For all $x \in K$, $F(x, x) = 0$ and $F(x, y)$ is $P-$convex in y; G is $Q-$concave and there exists $x_0 \in K$ such that $G(x_0) \subset corQ$.

We note that if G is $Q-$concave on K, then the set $N = \{x \in K : G(x) \subset Q\}$ is convex.

By using Theorem3.1 we present the following result which is a set-valued version of Theorem 3.1 in [2].

Theorem 3.2. *Suppose that the Condition C is satisfied and $corP \neq \emptyset$. If $x \in K$ is a solution of (GWVEP) then there exists $(p^*, q^*) \in P^* \setminus \{0\} \times (-Q)^*)$ such that $\langle q^*, G(x)\rangle = 0$ and*

$$\langle p^*, F(x, x)\rangle + \langle q^*, G(x)\rangle = \min_{y \in K} \left[\langle p^*, F(x, y)\rangle + \langle q^*, G(y)\rangle\right].$$

The converse is true when the range of G is a subset of Q, that is, $G(K) \subseteq Q$.

Theorem 3.3. *Assume that the Condition C is satisfied, and that P has a base B. Then $x \in N$ is a solution of (HGVEP) if and only if there exists $(p^*, q^*) \in P^{\Delta}(B) \times -Q^*$ such that $\langle q^*, G(x)\rangle = 0$ and*

$$\langle p^*, F(x, x)\rangle + \langle q^*, G(x)\rangle = \min_{y \in K} \left[\langle p^*, F(x, y)\rangle + \langle q^*, G(y)\rangle\right].$$

References

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