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CHARACTERIZATIONS OF h-CONVEX FUNCTIONS

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Abstract. In this paper, we study some characterizations of real valued h-convex functions.

1. Introduction

We say that [3] $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}\,.$$

For $s \in (0, 1]$, a function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex function, or that f belongs to the class K_s^2, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for every $x, y \in [0,\infty)$ and $t \in [0, 1]$, see [1]. Also, we say that $f : I \to [0,\infty)$ is a P-function [2], or that f belongs to the class $P(I)$, if for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y)\,.$$

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Throughout this paper, suppose that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined on J and I, respectively.

In [5], Varošanec defined the h -convex function as follows:
Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is a h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)\,. \tag{1.1}$$

If inequality (1.1) is reversed, then f is said to be h-concave, that is $f \in SV(h, I)$.

Obviously, if $h(t) = t$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$. If $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

A function $h : J \to \mathbb{R}$ is said to be a *super-additive function* if

$$h(x + y) \geq h(x) + h(y)\,, \tag{1.2}$$

for all $x, y \in J$. If inequality (1.2) is reversed, then h is said to be a *sub-additive function*. If the equality holds in (1.2), then h is said to be a *additive function*.

The function h is called a *super-multiplicative function* if

$$h(xy) \geq h(x)h(y)\,, \tag{1.3}$$

for all $x, y \in J$ [5]. If inequality (1.3) is reversed, then h is called a *sub-multiplicative function*. If the equality holds in (1.3), then h is called a *multiplicative function*.

Example 1.1. [5] Consider the function $h : [0, +\infty) \to \mathbb{R}$ by $h(x) = (c + x)^{p-1}$. If $c = 0$, then the function h is multiplicative. If $c \geq 1$, then for $p \in (0,1)$ the function h is super-multiplicative and for $p > 1$ the function h is sub-multiplicative.

2. Main results

Assume that C is a convex subset of a linear space X and f is an arbitrary real-valued function on C. The non-negative function $f : C \to \mathbb{R}$ is called *h-convex function* on C, if $f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$ for every $x, y \in C$ and $t \in [0,1]$.

Let x and y be two fixed elements in C. Define the map $f_{x,y}$ as follows:

$$f_{x,y} : [0,1] \to \mathbb{R}\,, \quad f_{x,y}(t) = f(tx + (1-t)y)\,.$$

The following theorem is a characterization of h-convex functions.

Theorem 2.1 (First characterization)**.** *With the above assumptions, the following statements are equivalent:*

(i) *f is a h-convex function on C.*
(ii) *The mapping $f_{x,y}$ is a h-convex function on $[0,1]$, for any $x, y \in C$.*

Now, for fixed $t \in [0,1]$, we define the function $f_t : C^2 \to \mathbb{R}$ by $f_t(x,y) = f(tx + (1-t)y)$.

In the next theorem, we state a new characterization of h-convex functions.

Theorem 2.2 (Second characterization)**.** *The following statements of h-convex functions hold:*

(i) *If f is a h-convex function on C, then f_t is a h-convex function on C^2 for every $t \in [0,1]$.*
(ii) *If C is a cone in X and f_t is a h-convex function on C^2 for every $t \in (0,1)$, then f is a h-convex function on C.*

Theorem 2.3 (Third characterization)**.** *Let h be a strictly positive multiplicative function, then the following statements are equivalent:*

(i) *f is a h-convex function.*
(ii) *If $(1+s)x - sy \in C$, for every $x, y \in C$ and $s \geq 0$, then*

$$f\big((1+s)x - sy\big) \geq h(1+s)f(x) - h(s)f(y)\,. \tag{2.1}$$

Theorem 2.4. (i) *Assume that X is a real vector space and $f : X \to \mathbb{R}$ is an even h-convex function. Then*

$$\begin{aligned}\frac{f((1-2t)x) + f((2t-1)y)}{h(t)+h(1-t)} &\leq f\big((1-t)x + ty\big) + f\big(tx + (1-t)y\big) \\ &\leq [h(t) + h(1-t)][f(x) + f(y)]\,. \end{aligned} \tag{2.2}$$

(ii) *Let X be a topological vector space, h be an integrable strictly positive function and f be a continuous even h-convex function, then*

$$\frac{1}{2}\int_0^1 [f(tx) + f(ty)]\ \mathrm{d}t \leq \int_0^1 [h(t) + h(1-t)] f\big(tx + (1-t)y\big)\ \mathrm{d}t\,. \tag{2.3}$$

In addition, if h is super-additive, then

$$\frac{1}{2h(1)\left(\int_0^1 h(t)\ \mathrm{d}t\right)} \int_0^1 [f(tx) + f(ty)]\ \mathrm{d}t \leq f(x) + f(y)\,. \tag{2.4}$$

Corollary 2.5. (i) *Assume that X is a real vector space and $f : X \to \mathbb{R}$ is an even convex function. Then*

$$\begin{aligned} f((1-2t)x) + f((2t-1)y) \leq & f\big((1-t)x+ty\big) + f\big(tx+(1-t)y\big) \\ \leq & f(x) + f(y)\,. \end{aligned} \tag{2.5}$$

(ii) *Let X be a topological vector space and f be a continuous even convex function, then*

$$\frac{1}{2}\int_0^1 [f(tx)+f(ty)]\,\mathrm{d}t \leq \int_0^1 f\big(tx+(1-t)y\big)\,\mathrm{d}t \leq f(x)+f(y)\,. \tag{2.6}$$

Proof. Enough put in Theorem 2.4, $h(t) = t$. □

Corollary 2.6. [4, Lemma 3.2]

(i) *Assume that X is a real vector space and $f : X \to \mathbb{R}$ is an even function in $P(I)$. Then*

$$\begin{aligned} \frac{f((1-2t)x) + f((2t-1)y)}{2} \leq & f\big((1-t)x+ty\big) + f\big(tx+(1-t)y\big) \\ \leq & 2\big(f(x) + f(y)\big)\,. \end{aligned} \tag{2.7}$$

(ii) *Let X be a topological vector space and f be a continuous even function in $P(I)$, then*

$$\frac{1}{4}\int_0^1 [f(tx)+f(ty)]\,\mathrm{d}t \leq \int_0^1 f\big(tx+(1-t)y\big)\,\mathrm{d}t \leq f(x)+f(y)\,. \tag{2.8}$$

Proof. In Theorem 2.4, put $h(t) = 1$. □

Example 2.7. [4, Theorem 3.3] Let $(X, \|\cdot\|)$ be a normed space, $x, y \in X$ and $0 < p < 1$. Since $f(x) = \|x\|^p$ is an even continuous P-convex function, we have the following Hermit-Hadamard inequality

$$\frac{\|x\|^p + \|y\|^p}{4(p+1)} \leq \int_0^1 \|(1-t)x+ty\|^p \mathrm{d}t \leq \|x\|^p + \|y\|^p\,.$$

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