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JENSEN-MERCER TYPE INEQUALITY FOR h-CONVEX FUNCTIONS

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Abstract. Mercer proved that if f is a convex function, then

$$f\Big(x_1 + x_n - \sum_{j=1}^{n} t_j x_j\Big) \leq f(x_1) + f(x_n) - \sum_{j=1}^{n} t_j f(x_j)\,.$$

where x_j's also satisfy in the condition $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, $t_j \geq 0$ and $\sum_{j=1}^{n} t_j = 1$. In this paper, we extend the Jensen-Mercer type inequality for real valued h-convex functions.

1. Introduction

We say that [3] $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}\,.$$

For $s \in (0, 1]$, a function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex function, or that f belongs to the class K_s^2, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

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for every $x, y \in [0, \infty)$ and $t \in [0, 1]$, see [1]. Also, we say that $f : I \to [0, \infty)$ is a P-function [2], or that f belongs to the class $P(I)$, if for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1 - t)y) \leq f(x) + f(y)\,.$$

Throughout this paper, suppose that I and J are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined on J and I, respectively.

In [5], Varošanec defined the h -convex function as follows:
Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is a h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $t \in (0, 1)$ we have

$$f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y)\,. \tag{1.1}$$

If inequality (1.1) is reversed, then f is said to be h-concave, that is $f \in SV(h, I)$.

Obviously, if $h(t) = t$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$. If $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.

A function $h : J \to \mathbb{R}$ is said to be a *super-additive function* if

$$h(x + y) \geq h(x) + h(y)\,, \tag{1.2}$$

for all $x, y \in J$. If inequality (1.2) is reversed, then h is said to be a *sub-additive function*. If the equality holds in (1.2), then h is said to be a *additive function*.

The function h is called a *super-multiplicative function* if

$$h(xy) \geq h(x)h(y)\,, \tag{1.3}$$

for all $x, y \in J$ [5]. If inequality (1.3) is reversed, then h is called a *sub-multiplicative function*. If the equality holds in (1.3), then h is called a *multiplicative function*.

Example 1.1. [5] Consider the function $h : [0, +\infty) \to \mathbb{R}$ by $h(x) = (c + x)^{p-1}$. If $c = 0$, then the function h is multiplicative. If $c \geq 1$, then for $p \in (0, 1)$ the function h is super-multiplicative and for $p > 1$ the function h is sub-multiplicative.

2. Main results

In [4], Mercer proved that

$$f\left(x_1 + x_n - \sum_{j=1}^{n} t_j x_j\right) \leq f(x_1) + f(x_n) - \sum_{j=1}^{n} t_j f(x_j)\,. \tag{2.1}$$

where x_j's also satisfy in the condition $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, $t_j \geq 0$ and $\sum_{j=1}^n t_j = 1$.

In this section, we present Jensen-Mercer inequality for h-convex functions.

Theorem 2.1. [5, Theorem 19] *Let $t_1, \cdots, t_n$ be positive real numbers $(n \geq 2)$. If h is a non-negative super-multiplicative function, f is a h-convex function on I and $x_1, \cdots, x_n \in I$, then*

$$f\left(\frac{1}{T_n}\sum_{j=1}^n t_j x_j\right) \leq \sum_{j=1}^n h\left(\frac{t_j}{T_n}\right) f(x_j)\,, \tag{2.2}$$

where $T_n = \sum_{j=1}^n t_j$.

Lemma 2.2. *Let $0 < x \leq y$ and f be a h-convex function, then for every $z \in [x, y]$, there exists $\lambda \in [0, 1]$ such that*

$$f(x + y - z) \leq [h(\lambda) + h(1-\lambda)][f(x) + f(y)] - f(z)\,. \tag{2.3}$$

Moreover, if h is super-additive, then

$$f(x + y - z) \leq h(1)[f(x) + f(y)] - f(z)\,.$$

Theorem 2.3. *Let f be a h-convex function on an interval containing the x_j $(j = 1, \cdots, n)$ such that $0 < x_1 \leq \cdots \leq x_n$, then*

$$\begin{aligned} &f\left(x_1 + x_n - \sum_{j=1}^n t_j x_j\right) \\ &\leq \left(\sum_{j=1}^n h(t_j)[h(\lambda_j) + h(1-\lambda_j)]\right)\big(f(x_1) + f(x_n)\big) - \sum_{j=1}^n h(t_j) f(x_j)\,, \end{aligned}$$

where for every $j = 1, \cdots, n$, there exists $\lambda_j \in [0, 1]$ such that $x_j = \lambda_j x_1 + (1-\lambda_j) x_n$.

Corollary 2.4. *With the assumptions of previous theorem, if h is a super-additive function such that for every probability vector $(t_1, \cdots, t_n)$, $\sum_{j=1}^n h(t_j) \leq 1$, then*

$$f\left(x_1 + x_n - \sum_{j=1}^n t_j x_j\right) \leq h(1)\big(f(x_1) + f(x_n)\big) - \sum_{j=1}^n h(t_j) f(x_j)\,.$$

Moreover, if h is multiplicative, then

$$f\left(x_1 + x_n - \sum_{j=1}^n t_j x_j\right) \leq f(x_1) + f(x_n) - \sum_{j=1}^n h(t_j) f(x_j)\,.$$

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