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THE EXISTENCE OF HYPERINVARIANT SUBSPACES FOR WEIGHTED SHIFT OPERATORS

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Abstract. We introduce some classes of Banach spaces for which the invariant subspace problem for the shift operator has positive answer. Moreover, we consider some cases of weighted spaces for which the problem remains open.

1. Introduction

Let X be a Banach space, and let $B(X)$ be the Banach algebra of all bounded linear operator on X. A closed subspace M of X is called an *invariant subspace* of an operator $A \in B(X)$ if $AM \subseteq M$. At the same token, it is called a *hyperinvariant subspace* of A, if it is invariant under every operator that commutes with A. In addition M is bi-invariant subspace of A if M is invariant subspace for A and A^{-1} whenever A is invertible. Throughout the paper, we assume that M is nontrivial and proper, i.e., $M \neq \{0\}$ and $M \neq X$. An old and still open problem in operator theory is the invariant subspace problem,that is whether every operator $A \in B(X)$ has non-trivial invariant subspace? In what follows, we consider this problem for shift operator in some weighted spaces.

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When β is a function from $\mathbb{Z}$ into $[0,\infty)$ set

$$\ell_\beta := \ell^2_\beta(\mathbb{Z}) = \left\{ u = (u_n)_{n\in\mathbb{Z}} : \|u\|^2_\beta = \sum_{n\in\mathbb{Z}} |u_n|^2 \beta^2(n) < \infty \right\}.$$

Here we denote by $\mathcal{H}(\mathbb{D})$ the space of functions holomorphic on the open unit disc $\mathbb{D}$, and for $f \in \mathcal{H}(\mathbb{D})$ we denote by $\widehat{f}(n)$ the n^{th} Taylor coefficient of f at the origin. Let $\sigma = \beta|\mathbb{Z}^+$, and denote by $H_\sigma := H^2_\sigma(\mathbb{D})$ the usual weighted Hardy space

$$\left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^2_\sigma := \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \sigma^2(n) < +\infty \right\}.$$

The usual shift operator on ℓ_β is the defined by the formula

$$S.u = (u_{n-1})_{n\in\mathbb{Z}}, \quad u = (u_n)_{n\in\mathbb{Z}} \in \ell_\beta.$$

Let

$$\begin{aligned}
\ell^+_\beta &= \{(u_n)_{n\in\mathbb{Z}} \in \ell_\beta : u_n = 0, n < 0\},\\
\ell^-_\beta &= \{(u_n)_{n\in\mathbb{Z}} \in \ell_\beta : u_n = 0, n \geq 0\},\\
S^+ &= S|\ell^+_\beta,\\
e_p &= (\delta_{p,n})_{n\in\mathbb{Z}}
\end{aligned}$$

where we denote by $\delta_{p,n}$ the Kronecker symbol. We can identify ℓ^+_β to $\ell^2_\sigma(\mathbb{Z}^+)$ in the obvious way, and the Fourier transform $f \to (\widehat{f}(n))_{n\geq 0}$ is an isometry from H_σ onto ℓ^+_β. Denote by $\check{u}$ the inverse Fourier transform, so that

$$\check{u}(z) = \Sigma_{n=0}^{\infty} u_n.z^n \text{ for } z \in \mathbb{D}, u \in \ell^+_\beta$$

Denote by Γ the unit circle of the complex plane , if $\beta(n) \geq 1$ for all $n \in \mathbb{Z}$ we have a subspace of $L^2(\Gamma)$ by the following definition.

$$L^2_\beta(\Gamma) = \left\{ f \in L^2(\Gamma) : \ \|f\|^2_\beta = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \beta^2(n) < \infty \right\}.$$

and

$$L^2_{\beta^-}(\Gamma) = \left\{ f \in L^2_\beta(\Gamma) \mid \hat{f}(n) = 0, \ \forall\ n \geq 0 \right\}.$$

In this case by using Fourier transform isometry we have $\ell^2_\beta(\mathbb{Z}) = \widehat{L^2_\beta(\Gamma)}$ and shift operator can be considerd as multiplicaton operator. We denote by $S : f(z) \to zf(z)$ the forward shift operator (multiplication by z) on $L^2_\beta(\Gamma)$, so for every $m,n \in \mathbb{Z}$ if $f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n) z^n$, then $\widehat{S^m f}(n) = \hat{f}(n-m)$.

The existence of nontrivial translation invariant subspaces of $\ell^2_\beta(\mathbb{Z})$ is an old-standing problem, and there were so far very few cases of concrete weights β for which translation invariant subspaces of $\ell^2_\beta(\mathbb{Z})$ have been classified. In 1932, the hyperinvariant subspaces of the shift operator on L^2 were characterized by Wiener [12] . According to his result, the reducing subspaces of the bilateral shift on $L^2(\Gamma)$ are precisely the subspaces $M = \{f \in L^2(\Gamma) :\ f(z) = 0 \ a.e. \text{ on } E\}$ for measurable subsets $E \subseteq \Gamma$. In 1949, A. Beurling [2] characterized the invariant subspaces of the shift operator on the Hardy space H^2. His result is one of the pillars of modern function theory and says that if M is an invariant subspace of the shift operator on the unit circle, then there exists an inner function ϕ on the unit circle such that $M = \phi H^2$.

Definition 1.1. A linear subspace M of H_β has the division property if $\frac{f(z)}{f(z)-\lambda}$ belongs to M for every $f \in M$ and every $\lambda \in Z(f)$.

J. Esterle and A. Volberg [8] introduced the class of weights $\mathcal{S}$ which consists of all weight β that satisfy the following conditions:

(1) $0 < \inf_{p\in\mathbb{Z}} \frac{\beta(n+p)}{\beta(p)} \leq \sup_{p\in\mathbb{Z}} \frac{\beta(n+p)}{\beta(p)} < \infty$.
(2) For all $n \geq 0$, if $\bar{\beta}(n) = \sup_{p>0} \frac{\beta(p)}{\beta(n+p)}$ and $\tilde{\beta}(n) = \sup_{p>0} \frac{\beta(n+p)}{\beta(p)}$, then $\lim_{n\to\infty},\ \bar{\beta}(n) = \lim_{n\to\infty} \tilde{\beta}(n) = 1$.

Then, using sharp estimates of Matsaev–Mogulskii about the rate of growth of quotients of analytic functions on the unit disc, they obtained the following result.

Theorem 1.2 (Esterle–Volberg)**.** *Let $\beta \in S$. If $\log \bar{\beta}_+(n) = O(n^\alpha)$ with $\alpha < 1/2$ and $\lim_{n\to\infty} \frac{\log \beta(n)}{\sqrt{n}} = \infty$, then $F = (\bigvee_{n\leq 0} S^n F) \cap L^+_\beta$ and $L^2_\beta(\Gamma) = (\bigvee_{n\leq 0} S^n F) + L^+_\beta$, for every closed subspace $F \neq 0$ of L^+_β having the division property.*

Definition 1.3. The weight β is called dissymmetric if

(1) $\beta(n) = 1, \qquad n \geq 1$,
(2) $\limsup\limits_{n\to-\infty} \frac{\beta(n-1)}{\beta(n)} < \infty$,
(3) $[\beta(n)]^{\frac{1}{|n|}} \to 1 \qquad \text{as } n \to -\infty$.

If U is a singular inner function, let $E(U)$ be the closure of $\bigvee\{S^n U : n \in \mathbb{Z}\}$ and $U^* = \frac{1}{U} - \frac{1}{\lim_{z\to\infty} U(z)}$.

J. Esterle [6] characterized the structure of bi-invariant subspaces of dissymmetric weighted shift. His main result is as follows.

Theorem 1.4 (Esterle)**.** *Let β be a dissymmetric weight such that*

$$\frac{\log \beta(-n)}{\sqrt{n}} \to \infty \qquad \text{as } n \to \infty.$$

Then the map $U \to E(U)$ is a bijection from the set of singular inner functions on $\mathbb{D}$ onto the set of proper closed shift bi-invariant subspaces F such that $F \cap H^2(\mathbb{D}) \neq \{0\}$. Moreover, if U is a singular inner function we have $E(U) \cap H^2(\mathbb{D}) = UH^2(\mathbb{D})$, $L^2_\beta(\Gamma) = E(U) + H^2(\mathbb{D})$, and $E(U) = \{f \in L^2_\beta(\Gamma) | fU^* = 0\}$.

Two more important results were obtained by A. Shields [10] and J. Wermer [11].

Theorem 1.5 (Shields)**.** *Let S be a bilateral weighted shift. Then the following assertions hold.*

(1) *If S is invertible, then the spectrum of S is the annulus*

$$[r(S^{-1})]^{-1} \leq |z| \leq r(S).$$

(2) *If S is not invertible, then the spectrum is the disc $|z| \leq r(S)$.*

Theorem 1.6 (Wermer)**.** *If the operator T is invertible with*

$$\sum_{n\in\mathbb{Z}} \frac{\log \|T^n\|}{1+n^2} < \infty,$$

and spectrum of T contains more than one point, then T has nontrivial hyperinvariant subspaces.

In what follows, the existence and structure of hyperinvariant subspaces for the weighted shift operator will be investigated. We consider two types of weights:

$$\omega(n) = \begin{cases} 1 & \text{if } n > 0, \\ \exp(|n|^\alpha) & \text{if } n \leq 0, \end{cases}$$

and

$$\tau(n) = \begin{cases} 1 & \text{if } n > 0, \\ \exp(-|n|^\alpha) & \text{if } n \leq 0, \end{cases}$$

where ($\alpha \in \mathbb{R}^+$). It is elementary to see that ω is log-concave if $\alpha \leq 1$, and that τ is log-concave if $\alpha \geq 1$. In what follows, β denotes either ω or τ, and α denotes the power of $|n|$ in $\exp(|n|^\alpha)$.

2. Main results

In what follows, we state some theorems which enable us to solve the hyper invariant subspace problem in some cases.

Theorem 2.1. *The shift operator on $\ell^2_\tau(\mathbb{Z})$ is invertible for every positive α and invertible on $\ell^2_\omega(\mathbb{Z})$ if and only if $\alpha \leq 1$.*

Proof. For each $u \in \ell^2_\beta(\mathbb{Z})$, if $u = (u_n)_{n\in\mathbb{Z}}$, then $S^{-1}u = (u_{n+1})_{n\in\mathbb{Z}}$. So for every positive weight β we have

$$\begin{aligned} \|S^{-1}u\|^2_\beta &= \sum_{n\in\mathbb{Z}} |u_n|^2\beta^2(n-1) \\ &= \sum_{n\in\mathbb{Z}} |u_n|^2\beta^2(n)\left(\frac{\beta(n-1)}{\beta(n)}\right)^2. \end{aligned} \tag{2.1}$$

Let $\beta = \tau$. we get

$$\left(\frac{\beta(n-1)}{\beta(n)}\right) = \begin{cases} 1 & n > 0 \\ \exp(|n|^\alpha - (|n|+1)^\alpha) & n \leq 0, \end{cases}$$

which is bounded for every positive α. Similarly, when $\beta = \omega$ we get

$$\left(\frac{\beta(n-1)}{\beta(n)}\right) = \begin{cases} 1 & n > 0 \\ \exp((|n|+1)^\alpha - |n|^\alpha) & n \leq 0, \end{cases}$$

that is bounded if and only if $\alpha \leq 1$. Therefore by using (2.1) we have

$$\|S^{-1}u\|^2_\beta \leq M \sum_{n\in\mathbb{Z}} |u_n|^2\beta^2(n) = M\|u\|^2_\beta,$$

for some positive M. □

Lemma 2.2. *If $\alpha > 1$ and $1 \leq k \leq n-1$, then for sufficiently large n,*

$$(n-1)\exp(-n^\alpha) \leq \exp(-k^\alpha - (n-k)^\alpha). \tag{2.2}$$

Proof. The inequality (2.2) is equivalent to $\ln(n-1) \leq n^\alpha - k^\alpha - (n-k)^\alpha$. Put $f(k) = k^\alpha + (n-k)^\alpha$ for $k = 1, ..., n-1$.
Note that f is a concave function and takes its maximum in 1 or $n-1$.
Therefore,

$$n^\alpha - k^\alpha - (n-k)^\alpha \geq n^\alpha - (n-1)^\alpha - 1. \tag{2.3}$$

On the other hand, there exists $t \in [n-1, n]$ such that

$$n^\alpha - (n-1)^\alpha = \alpha t^{\alpha-1}.$$

Hence

$$n^\alpha - (n-1)^\alpha \geq \alpha(n-1)^{\alpha-1}. \tag{2.4}$$

Since

$$\lim_{n\to\infty}\frac{\alpha(n-1)^{\alpha-1}}{\ln(n-1)+1}=\lim_{n\to\infty}\alpha(\alpha-1)(n-1)^{\alpha-1}=\infty.$$

we have

$$\alpha(n-1)^{\alpha-1}\geq \ln(n-1)+1, \tag{2.5}$$

for all sufficiently large n.
So by using equations (2.3), (2.4) and (2.5) we have

$$\ln(n-1)\leq n^{\alpha}-(n-1)^{\alpha}-1\leq n^{\alpha}-k^{\alpha}-(n-k)^{\alpha}.$$

□

Theorem 2.3. *The following properties hold for weighted spaces $\ell^2_{\tau}(\mathbb{Z})$ and $L^2_{\omega^-}(\Gamma)$.*

1. *Let $u,v\in \ell^2_{\tau^-}(\mathbb{Z})$ and $\alpha>1$. Then $u*v\in \ell^2_{\tau^-}(\mathbb{Z})$, i.e., $\ell^2_{\tau^-}(\mathbb{Z})$ is stable under convolution.*
2. *If $\alpha>0$, then $L^2_{\omega^-}(\Gamma)\oplus H^{\infty}\subseteq L^{\infty}(\Gamma)$.*

Proof. (1) Suppse $u,v\in L^2_{\tau^-}(\Gamma)$, then we have

$$\sum_{n<-1}|(u*v)_n|^2\tau^2(n)=\sum_{n<-1}\tau^2(n)\left(\sum_{p=1}^{|n|-1}u_{-p}v_{n+p}\right)^2.$$

Let $\max\{u_{-p}v_{n+p}:\ 1\leq p\leq |n|-1\}=u_{-k}v_{n+k}$. By using Lemma 2.2 it is easy to see that

$$\begin{aligned}
&\sum_{n<-1}\tau^2(n)\left(\sum_{p=1}^{|n|-1}u_{-p}v_{n+p}\right)^2\\
&\quad\leq \sum_{n<-1}\tau^2(n)\left(u_{-k}v_{n+k}(|n|-1)\right)^2\\
&\quad\leq \left(\sum_{n<-1}\tau^2(-k)|u_{-k}|^2\tau^2(n+k)|v_{n+k}|^2\right)+C\\
&\quad\leq C'\|u\|^2_{\tau}\|v\|^2_{\tau},
\end{aligned}$$

Hence (1) is proved.

(2) If $f\in L^2_{\omega^-}(\Gamma)$ and $z\in\Gamma$, then $f(z)=\sum_{n<0}\hat{f}(n)z^n$.

Therefore

$$
\begin{aligned}
|f(z)| &\le \sum_{n<0} |\hat{f}(n)| = \sum_{n<0} |\hat{f}(n)| \frac{\omega(n)}{\omega(n)} \\
&\le \left(\sum_{n<0} |\hat{f}(n)|^2 \omega^2(n) \right)^{\frac{1}{2}} \left(\sum_{n<0} \frac{1}{\omega^2(n)} \right)^{\frac{1}{2}} \\
&= \|f\|_\omega^2 \sum_{n>0} \frac{1}{\exp n^\alpha} < \infty.
\end{aligned}
$$

□

In what follows, we state a new theorems which is very useful for solving the hyper invariant subspace problem in some cases.

Theorem 2.4. *Let S be the shift operator on either $\ell^2_\tau(\mathbb{Z})$ or $\ell^2_\omega(\mathbb{Z})$. Then the following statements hold.*

(1) *If $\alpha > 1$, then S is unbounded on $\ell^2_\tau(\mathbb{Z})$.*
(2) *If $0 < \alpha \le 1$, then S is bounded on $\ell^2_\tau(\mathbb{Z})$ and $\|S^n\|_\tau = \tau(-n)$ for every $n \in \mathbb{Z}^+$.*
(3) $r(S) = \begin{cases} 1 & \alpha < 1 \\ e & \alpha = 1 \end{cases}$ *on $\ell^2_\tau(\mathbb{Z})$ and* $\sum\limits_{n<0} \frac{\log \|(S^{-1})^n\|_\tau}{1+n^2} = \sum\limits_{n>0} \frac{n^\alpha}{1+n^2}$.
(4) $r(S) = 1$ *on $\ell^2_\omega(\mathbb{Z})$, and* $\sum\limits_{n>0} \frac{\log \|S^n\|_\omega}{1+n^2} = 0$.

Proof. (1) Assume $\alpha > 1$, for $n \in \mathbb{N}$ put $U^{(n)} = (\exp(|n|^\alpha)\delta_{p,-n})_{p\in\mathbb{Z}}$. It is obvious that $U^{(n)} \in \ell^2_\tau(\mathbb{Z})$ with $\|U^{(n)}\|_\tau = 1$ and $\|SU^{(n)}\|^2_\tau = \|\left(\exp(|n|^\alpha)\delta_{p,-n+1}\right)_{p\in\mathbb{Z}}\|^2_\tau$.

So

$$
\|SU^{(n)}\|^2_\tau = \exp 2(|n|^\alpha - |n-1|^\alpha),
$$

On the other hand, for each $n \in \mathbb{N}$ there exists $t \in [n-1, n]$ such that $|n|^\alpha - |n-1|^\alpha = \alpha |t|^{\alpha-1}$, thus $|n|^\alpha - |n-1|^\alpha \to \infty$ as $n \to \infty$. Therefore in the case $\alpha > 1$ we have $\|S\|_\tau = \infty$.

(2) For each $k \in \mathbb{N}$ and $u \in \ell^2_\tau(\mathbb{Z})$ we have $S^k u = (u_{n-k})_{n\in\mathbb{Z}}$.

Therefore

$$\begin{aligned}\|S^k u\|_\tau^2 &= \sum_{n\in\mathbb{Z}} |u_{n-k}|^2 \tau^2(n) \\ &= \sum_{n<0} |u_{n-k}|^2 \exp\left(-2|n-k|^\alpha + 2(|n-k|^\alpha - |n|^\alpha)\right) \\ &\quad + \sum_{n\geq 0} |u_{n-k}|^2.\end{aligned} \tag{2.6}$$

Consider that $\alpha \leq 1$, therefore $(n+k)^\alpha - n^\alpha$ is decreasing on $(0,\infty)$.
So this is obvious that

$$(n+k)^\alpha - (n)^\alpha \leq k^\alpha. \tag{2.7}$$

Hence by using (2.6), and (2.7), we have

$$\begin{aligned}\|S^k u\|_\tau^2 &\leq \exp(2k^\alpha)\left(\sum_{n<0} |u_{n-k}|^2 \exp(-2|n-k|^\alpha)\right) \\ &\quad + \sum_{0\leq n<k} |u_{n-k}|^2 + \sum_{i\geq 0} |u_i|^2 \\ &\leq \exp(2k^\alpha)\left(\sum_{i<-k} |u_i|^2 \exp(-2|i|^\alpha) + \sum_{-k\leq i<0} |u_i|^2 \exp(-2|i|^\alpha)\right) \\ &\quad + \sum_{i\geq 0} |u_i|^2 \\ &\leq \exp(2k^\alpha)\|u\|_\tau.\end{aligned}$$

Therefore for each $k \in \mathbb{N}$ we have

$$\|S^k\|_\tau \leq \exp(k^\alpha). \tag{2.8}$$

Moreover, if $u = (\exp(|k|^\alpha)\delta_{p,-k})_{p\in\mathbb{Z}}$, then $\|u\|_\tau = 1$ and $\|S^k u\|_\tau = \exp(k^\alpha)$.
So by (2.8) we proved (2).
(3) It is obvious from (2) that for every $k \in \mathbb{N}$ we have

$$\|S^k\|_\tau = \exp k^\alpha, \quad \text{and so} \quad r(S) = \begin{cases} 1 & \alpha < 1 \\ e & \alpha = 1. \end{cases}$$

In addition

$$\sum_{n<0} \frac{\log \|(S^{-1})^n\|_\tau}{1+n^2} = \sum_{n>0} \frac{\log \|S^n\|_\tau}{1+n^2} = \sum_{n>0} \frac{n^\alpha}{1+n^2}.$$

(4) Let S be the shift operator on $\ell^2_\omega(\mathbb{Z})$.

If $u \in \ell^2_\omega(\mathbb{Z})$ and $u = (u_n)_{n\in\mathbb{Z}}$, then

$$S(u) = (u_{n-1})_{n\in\mathbb{Z}}.$$

So

$$\|Su\|^2_\omega = \sum_{n\in\mathbb{Z}} |u_n|^2\omega^2(n+1) = \sum_{n\in\mathbb{Z}} |u_n|^2\omega^2(n)\left(\frac{\omega(n+1)}{\omega(n)}\right)^2.$$

Since $\omega(n)$ is nonincreasing we get $\frac{\omega(n+1)}{\omega(n)} \le 1$, which implies $\|S\|_\omega \le 1$. On the other hand, $\|S^n(\delta_{p,1})_{p\in\mathbb{Z}}\|_\omega = 1$ for every $n \in \mathbb{Z}^+$, Hence it is obvious that $\|S^n\|_\omega = 1$.

It is then easily verified that

$$\sum_{n>0} \frac{\log\|S^n\|_\omega}{1+n^2} = 0, \quad \text{and} \quad r(S) = 1 \text{ on } L^2_\omega(\Gamma).$$

This completes the proof. □

Remark 2.5. By the similar argument as in Theorem 2.4, if $\alpha \le 1$, then $\sum_{n>0} \frac{\log\|(S^{-1})^n\|_\omega}{1+n^2} = \sum_{n>0} \frac{n^\alpha}{1+n^2}$. In addition we have $\|(S^{-1})^k\|_\omega = 1$ for every $k \in \mathbb{Z}^-$.

Thus when S be the shift operator on $\ell^2_\tau(\mathbb{Z})$, Theorem 1.5 implies that

$$\sigma(S) = \begin{cases} \Gamma & \alpha < 1 \\ \text{annulus } \ 1 \le |z| \le e & \alpha = 1 \end{cases}$$

and if consider S as shift operator on $\ell^2_\omega(\mathbb{Z})$, then

$$\sigma(S) = \begin{cases} \Gamma & \alpha < 1 \\ \text{annulus } \ e^{-1} \le |z| \le 1 & \alpha = 1 \end{cases}.$$

By Theorems 1.5, 1.6, 2.4 and Remark 2.5 we get immediately the following theorem.

Theorem 2.6. *If $0 \le \alpha < 1$ the hyper- invariant subspace problem for the shift operator on $\ell^2_\beta(\mathbb{Z})$ has positive answer.*

Now we are in the situation that we can assert the following theorems.

Theorem 2.7. *Let $0 < \alpha < 1$ and F be a left invariant subspace of shift operator in $L^2_\omega(\Gamma)$. If $F \cap L^2_{\omega^+}(\Gamma) \ne 0$, then F has division property.*

Proof. It is obvious by using [7, Proposition 3.2]. □

Theorem 2.8. *Let F be a left invariant subspace of shift operator in $L^2_\omega(\Gamma)$ and $1/2 < \alpha < 1$. If $L^2_{\omega+}(\Gamma) \cap F \neq \{0\}$, then $L^2_\omega(\Gamma) = F + L^2_{\omega+}(\Gamma)$. If F is also biinvariant, then $F = E(U)$ for some singular inner function.*

Proof. It follows from Theorems 1.2, 1.4, and 2.7. □

Since $L^2_\omega(\Gamma) \subseteq L^2(\Gamma)$ every invariant subspace of the shift operator in $L^2_\omega(\Gamma)$ is again an invariant subspace for S in $L^2(\Gamma)$. When $\sum_{n<0} \frac{\log \beta(n)}{n^2} < \infty$, with using the discrete version of the Beurling–Malliavin theorem [3] and J. Wermer [12], if M be a nontrivial hyperinvariant subspace of $L^2(\Gamma)$ for S, then $M \cap L^2_\beta(\Gamma) \neq \{0\}$.This leads to a presentation of a family of biinvariant subspaces for weighted shift operator.

When $\sum \frac{\log \beta(n)}{n^2} = \infty$, the existence and the structure of hyperinvariant subspaces for shift operator is an open problem. For more details about this condition see [6] and [8]. In this case Esterle and Volberg [8] proved the following theorem.

Theorem 2.9. *Let $\beta \in \mathcal{S}$ be a weight satisfying the following conditions.*

(1) $\sum_{n<0} \frac{\log \beta(n)}{n^2} = \infty$;
(2) $\left(\frac{\log \beta(-n)}{n}(\log n)^a\right)_{n\geq 0}$ *is eventually increasing for some* $a > 0$;
(3) $(\beta(-n)/n^\alpha)_{n\geq 0}$ *is eventually log-concave for some* $\alpha > 3/2$;
(4) $\limsup_{n\to\infty} \frac{\log \bar{\beta}^+(n)}{\log \beta(n)} < 1/200$.

Then for every $u \in L^2_\beta(\Gamma)$ there exists $v \in L^2_{\beta+}(\Gamma)$ and $k \geq 0$ such that

$$\bigvee_{n\leq 0} S^n u = \bigvee_{n\leq -k} S^n v,$$

and for every nontrivial left-invariant subspace M of ℓ_β, there exists $k \geq 0$ and a closed subspace N of $H_{\beta+}$ having the division property such that $M = \bigvee_{n\leq -k} S^n N$.

References

1. A. Atzmon, *Operators which are annihilated by analytic functions and invariant subpaces*, Acta Math. **144** (1980), 27–63.
2. A. Beurling, *On two problems concerning linear transformations on Hilbert spaces*, Acta. Math. **81** (1949), 239-255.
3. A. Beurling, *Mittag-Leffler Lectures in Complex Analysis*, Collected works of A. Beurling, (1977-78), pp. 361-443.

4. A. Beurling and P. Malliavin, *On Fourier transforms of measures with compact support*, Acta Math.**107** (1962), 291-309.
5. I. Chalendar and J. Partington, *Modern Approaches to the Invariant-Subspace Problem*, Cambridge University Press, 2011.
6. J. Esterle, *Singular inner functions and invariant subspaces for dissymetric weighted shifts*, J. Funct. Anal. **144** (1997), 64–104.
7. J. Esterle and A. Volberg, *Analytic left-invariant subspace of weighted Hibert spaces of sequence*, J. Operator Theory.**45** (2001), 265–301.
8. J. Esterle and A. Volberg, *Asymptotically holomorphic functions and translation invariant subspaces of weighted Hilbert spaces of sequences*, Annales Scientifiques de l'Ecole Normale Supérieure. **35** (2002) 185-230.
9. H. Radjavi and P. Rosenthal, *Invariant Subspaces*,Springer-Verlag, NewYork, 1973.
10. A. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Amer. Math. Soc. Math. Surveys, Vol. 13 ,1974, pp. 49 -128.
11. J. Wermer, *The existence of invariant subspaces*, Duke Math. J. **19** (1952), 615-622.
12. N. Wiene,*Tauberian theorems*, Ann. Math.**33**(1932), 1-100.
13. B.S. Yadav,*The Present State and Heritages of the Invariant Subspace Problem*, Milan J. Math. **73** (2005), 289-316.