The Extended Abstracts of
The 6^{th} Seminar on Functional Analysis and its Applications
4-5th March 2020, University of Isfahan, Iran

VALUE FUNCTION FOR NON-CONVEX VARIATIONAL PROBLEMS

MARZIEH DARABI

[1] *Department of Basic sciences, Golpayegan University of Technology*
marzi.darabi@yahoo.com, darabi@gut.ac.ir

Abstract. We consider non-convex variational problems. We improve the existence results for feasible solutions and obtain a necessary and sufficient condition for the existence solution of these problems.

1. Introduction

Non-convex variational problems have played a crucial role in the mathematical economics. One of the most important subjects in economic observations is to analyze the relationship between traders and productions. In this paper, our purpose is to find the total cost of a material that is purchased by a trader. In this paper, we obtain some sufficient and necessary conditions for existence of a feasible solution of the non-convex variational problem and we define optimal value function corresponding to the non-convex variational problem and obtain a relationship between subdifferential of the optimal value function and the set of Lagrange multipliers.

Let X, Y and Z be Banach spaces, K and W be closed, convex and pointed cones in Y and Z, respectively. We denote the positive polar cone of K by K^+ as

$$K^+ = \{y^* \in Y^* : \ \langle y^*, y\rangle \geq 0 \ \forall y \in K\}.$$

2010 *Mathematics Subject Classification.* 49J15; 49K15; 49J52.
Key words and phrases. Non-convex variational problems; Hamiltonian.

Suppose $\Gamma : [0,1] \longrightarrow 2^X$ is a set-valued mapping such that for all $t \in [0,1]$, $\Gamma(t)$ is a nonempty and closed set and

$$\mathcal{Z} = \{\gamma \in L^1([0,1], X) : \ \gamma(t) \in \Gamma(t) \ a.e. \ t \in [0,1]\}.$$

Suppose that $f_0 : [0,1] \times X \longrightarrow Y$ and $g_0 : [0,1] \times X \longrightarrow Z$ are Bochner integrable functions. Let us define the maps $f : L^1([0,1], X) \longrightarrow Y$ and $g : L^1([0,1], X) \longrightarrow Z$ as

$$f(\gamma) = \int_0^1 f_0(t, \gamma(t))dt,$$

$$g(\gamma) = \int_0^1 g_0(t, \gamma(t))dt.$$

We consider the following optimization problem

$$\min_{\gamma \in K(a)} f(\gamma) \ (OP(a))$$

where, for all $a \in Z$, we define $K(a) = \{\gamma \in \mathcal{Z} : \ g(\gamma) \in -W + a\}$. We denote the solution set of Problem $(OP(a))$ by $S(a)$. The Problem $(OP(a))$ is a generalization of the Problem (1.1) in [1], the Problem (P) in [3], the Problem $(\mathcal{P})$ in [6] and the Problem (P) in [7] . Hence, the amount of the material j to be purchased by the trader t is denoted by $\gamma_j(t)$. Therefore, the Problem $(OP(a))$, means the total cost of set $K(a)$ that consists of materials γ. Our goal is to obtain a solution for Problem $(OP(a))$ in Banach spaces where the functions $f_0(., \gamma(.))$ and $g_0(., \gamma(.))$ are Bochner integrable. For undefined notions we refer to [4].

Definition 1.1. A Set-valued operator $T : X \longrightarrow 2^Y$ is called:
(a) closed if $\mathrm{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.
(b) intersectionally closed on $A \subseteq X$, if;

$$\bigcap_{x \in A} cl\big(T(x)\big) = cl\big(\bigcap_{x \in A} T(x)\big).$$

(c) topological pseudomonotone, if for all $a, b \in Y$,

$$cl\big(\bigcap_{u \in [a,b]} T(u)\big) \cap [a, b] = \bigcap_{u \in [a,b]} T(u) \cap [a, b],$$

where, $[a, b] = \{x \in Y : \ a \leq_K x \leq_K b\}$.
(d) KKM map, if

$$\mathrm{conv} H \subseteq \bigcup_{x \in H} T(x), \text{ for each } H \in \langle X \rangle,$$

where, we denote by $\langle X \rangle$ the family of all nonempty finite subsets of the set X.

Definition 1.2 (Definition 3.6.1 in [2]). Let X be a Banach space and A be a nonempty subset of X. The Normal cone of K is defined as

$$N(A,a)=\begin{cases} inf\{v\in X^*:\ \langle v,x-a\rangle\leq 0\ \forall x\in A\} & \text{if } a\in A,\\ \\ \emptyset & \text{o.w,}\end{cases}$$

Theorem 1.3 (Theorem 2 in [5]). *Let K be a nonempty and convex subset of a Hausdorff topological vector space X and $T:K\longrightarrow 2^K$. Suppose that the following conditions hold:*
(A1) T is a KKM map;
(A2) for each $H\in\langle K\rangle$, the set-valued map $T\cap \mathrm{conv}H$ is intersectionally closed on $\mathrm{conv}H$;
(A3) T is topological pseudomonotone;
(A4) there exist a nonempty subset B of K and a nonempty compact subset D of K such that $\mathrm{conv}(H\cup B)$ is compact, for any $H\in\langle K\rangle$, and for each $y\in K\setminus D$, there exists $x\in\mathrm{conv}(B\cup\{y\})$ such that $y\notin T(x)$.
Then, $\bigcap_{x\in K}T(x)\neq\emptyset$.

2. Main results

In this section, we consider sufficient conditions for the existence of a solution of Problem $(OP(a))$. In the following, we obtain an existence result of Problem $(OP(a))$ by using a fixed point theorem.

Theorem 2.1. *Suppose that the set-valued map $T:K(a)\cap(L^1([0,1],X))\longrightarrow 2^{K(a)\cap(L^1([0,1],X))}$ defined as*

$$T(\gamma)=\{\alpha\in L^1([0,1],X)\cap K(a):\ f(\gamma)-f(\alpha)\in K\}.$$

(a) $K\subseteq Y$, is a pointed convex cone such that $K\cup -K=Y$;
(b) f_0 is a lower semi continuous map and K-convex in the second argument;
(c) there exist a nonempty subset B of $(K(a)\cap L^1([0,1],X))$ and a nonempty compact subset D of $(K(a)\cap L^1([0,1],X))$ such that $\mathrm{conv}(H\cup B)$ is compact, for any $H\in\langle K\rangle$, and for each $\alpha\in K\setminus D$ there exists $\gamma\in\mathrm{conv}(B\cup\{\alpha\})$ such that $\alpha\notin T(\gamma)$.
Then, Problem $(OP(a))$ has a solution.

Bazan and etal in [1], showed that if f_0 and g_0 are measurable, lower semi-continuous and continuous and W is a closed convex cone, then $0\in int[g(C_0)+W]$ and each local efficient solution for $(OP(a))$ is a global efficient solution. Here, we shall obtain by different assumptions that each local efficient solution of $(OP(a))$ is a global efficient solution.

Theorem 2.2. *Let f_0 be a lower semi continuous function and f_0 and g_0 be K-convex functions in the second argument. Then each local efficient solution of $(OP(a))$ is a global efficient solution.*

Here, we consider sufficient conditions for the existence of a feasible solution of Problem $(OP(a))$.

Definition 2.3. Let $\eta : X \times X \longrightarrow X$.

- [8] A subset Ω of X is said to be invex with respect to η if, for any $x, y \in \Omega$ and $\lambda \in [0,1]$, $\; y + \lambda\eta(x,y) \in \Omega$.
- [10] Let $\Omega \subset X$ be an invex set with respect to η and $F : \Omega \longrightarrow 2^Y$. F is said to be K-preinvex with respect to η on Ω if for any $x_1, x_2 \in \Omega$ and $\lambda \in [0,1]$, one has

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x_2 + \lambda\eta(x_1,x_2)) + K.$$

Definition 2.4. [12] Given a subset A of a Banach space Y which is ordered by a closed and convex cone $K \subset Y$, we say that $\bar{a} \in A$ is a super minimal point of A $\; (\bar{a} \in SE(A;K))$ if there is a number $M > 0$ such that

$$cl[cone(A - \bar{a})] \cap (B_Y - K) \subset MB_Y,$$

where B_Y denotes the closed unit ball of Y. Let $(\bar{x}, \bar{y}) \in grF$ with $\bar{x} \in \Omega$, then $(\bar{x}, \bar{y})$ is a local super minimmizer of problem

$$\text{minimize } F(x), \text{ subject to } x \in \Omega,$$

if there is a neighbourhood U of $\bar{x}$ such that $\bar{y} \in SE(F(\Omega \cap U);K)$. Obviously, every local super efficient solution of the above problem is a local efficient solution.

Theorem 2.5. [9] *Let Ω be a closed and invex set, K be a closed convex pointed ordering cone in Y and $F : \Omega \longrightarrow 2^Y$ be a K-preinvex map with respect to η which is continuous with respect to the second argument. Suppose that $(\bar{x}, \bar{y}) \in GrF$ and there is a $y^* \in intK^+$ such that*

$$0 \in \partial F(\bar{x}, \bar{y})(y^*) + N(\bar{x}; \Omega).$$

Then $(\bar{x}, \bar{y})$ is a local super minimizer of Problem

$$\text{minimize } F(x), \text{ subject to } x \in \Omega.$$

Theorem 2.6. *Let K be a closed convex pointed ordering cone in Y and $f : L^1([0,1], \mathbb{R}^n)) \longrightarrow Y$ be a convex map. Suppose that there is a $y^* \in intK^+$ such that*

$$0 \in \partial f(\bar{\gamma})(y^*) + N(\bar{\gamma}; L^1([0,1], \mathbb{R}^n)).$$

Then $\bar{\gamma}$ is a local super minimizer of Problem $(OP(a))$.

Proof. Obviously; $L^1([0,1],\mathbb{R}^n)$ is a closed and convex set and therefore, it is invex with respect to η which $\eta(x,y) = x - y$. Now, an appeal to Theorem 2.5 completes the proof. □ □

For Problem $(OP(a))$ and fixed ${y_0}^* \in K^+$, the optimal value function $\psi : Z \longrightarrow \mathbb{R}$ is defined as:

$$\psi(a) = \begin{cases} inf\{{y_0}^*(f(\gamma)) : \ g(\gamma) \in -W + a\} & K(a) \neq \emptyset, \\ \\ +\infty & \text{o.w}, \end{cases}$$

Obviously, if there exists $\gamma_0 \in K(a)$ such that $\psi(a) = {y_0}^*(f(\gamma_0))$, then $\gamma_0 \in S(a)$ and the converse holds as well. In the following results, we focus on the set of Lagrange multipliers $f(\gamma) + \langle\lambda, g(\gamma)\rangle$, where $\lambda \in Z^*$ and $\gamma \in L^1([0,1], X)$. We shall show a relation between subdifferential of the optimal value function for Problem $(OP(0))$ and the set of Lagrange multipliers which refines the Corollary 3.7 in [1]. Motivated by an idea in [12], we consider the following result.

Theorem 2.7. *$-\lambda \in \partial\psi(0)$ if and only if*
(i) $\lambda \in W^+$;
(ii) for all $\gamma \in K(a)$,

$$\psi(0) \leq {y_0}^*(f(\gamma)) + \langle\lambda, g(\gamma)\rangle.$$

In the following theorem, we obtain a relationship between solutions of Problem $(OP(0))$ and the set of Lagrange multipliers.

Theorem 2.8. *Let $(\bar{x}, \lambda) \in X \times Z^*$ then, $-\lambda \in \partial\psi(0)$ and $\bar{x} \in S(0)$, if and only if*
(i) $\lambda \in W^+$;
(ii) $\bar{x}$ is a solution of the following problem (I)

$$\min_{\gamma \in K(a)} ({y_0}^*(f(\gamma)) + \langle\lambda, g(\gamma)\rangle) \ (I).$$

(iii) $\langle\lambda, g(\bar{x})\rangle = 0$.

Now, we consider the Problem $(OP(a))$. For this idea, we define the Hamiltonian function $H : [0,1] \times Z^* \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ corresponding to Problem $(OP(a))$ as

$$H(t,\lambda) = \sup_{\gamma \in \mathfrak{Z}}\{\langle\lambda, g(\gamma)\rangle - {y_0}^*(f(\gamma))\}.$$

In the following theorem, we will show the existence of a solution of the Problem $(P(a))$ is equivalent to

$$H(t,\lambda) = \sup_{\gamma \in \mathfrak{Z}}\{\langle\lambda, g(\gamma)\rangle - {y_0}^*(f(\gamma))\},$$

and we will obtain a relationship between the subdifferential of the optimal value function of Problem ($P(a)$) and the set of Lagrange multipliers. In fact,we obtain a necessary and sufficient condition for the existence of solution of Problem ($P(a)$).

Theorem 2.9. *If $\lambda \in \partial\psi(a)$ and $\bar{x} \in S(a)$, then*
(i) $-\lambda \in W^+$;
(ii) $\lambda \in N(W, -g(\bar{x}) - a)$ *and*

$$H(t, \lambda) = {y_0}^*(f(\bar{x})) - \langle \lambda, g(\bar{x}) \rangle.$$

Conversely, if $-\lambda \in W^+$, $\lambda \in N(W, -g(\bar{x}) + a)$ *and*

$$H(t, \lambda) = {y_0}^*(f(\bar{x})) - \langle \lambda, g(\bar{x}) \rangle,$$

then $\lambda \in \partial\psi(a)$ *and* $\bar{x} \in S(a)$.

The following result is a generalization of Theorem 4.1 in [1].

Corollary 2.10. $\lambda \in \partial\psi(0)$ *and* $\bar{x} \in S(0)$, *if and only if*
(i) $-\lambda \in W^+$;
(ii) $\lambda \in N(W, -g(\bar{x}))$ *and*

$$H(t, \lambda) = \langle \lambda, g(\bar{x}) \rangle - {y_0}^*(f(\bar{x})).$$

Proof. It suffices to set $a = 0$ in the Theorem 2.9. □ □

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