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A FAN-TYPE SECTION THEOREM IN UNIFORM MAPCONVEX SPACES

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Abstract. In this paper, we present a Fan-type section Theorem. Here, the result is investigated in some topological spaces which do not necessarily have a linear structure.

1. Introduction

In 1961, Ky Fan [1] proved the following section theorem:

Theorem 1.1. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E, and A be a nonempty subset of $X \times X$ with $(x, x) \in A$ for all $x \in X$. Suppose that the following conditions are satisfied:*

(i) *For each $x \in X$, the set $\{y \in X : (x, y) \notin A\}$ is convex or empty.*
(ii) *For each $y \in X$, the set $\{x \in X : (x, y) \in A\}$ is closed in X.*

Then, there exists a point $x_0 \in X$, such that $\{x_0\} \times X \subset A$.

Later he stated an equivalent form of this geometric theorem in [2] as follows:

Theorem 1.2. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and let $A \subseteq X \times X$. Assume*

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(i) *For each* $y \in X$, *the set* $\{x \in X : (x, y) \notin A\}$ *is convex and nonempty.*
(ii) *For each* $x \in X$, *the set* $\{y \in X : (x, y) \in A\}$ *is open in* X.

Then, there exists a point $x_0 \in X$, *such that* $(x_0, x_0) \in A$.

The above theorem has numerous applications in various fields of mathematics and unifies many results in the literature, for instance, minimax inequality, fixed point theorems for set-valued maps, mathematical economics and game theory, see for example [3, 4, 5] and references therein.

In this paper we establish a Fan-type section Theorem. This result is presented in uniform mapconvex spaces.

Here $\langle X \rangle$ denotes the set of all finite subsets of X and $\triangle_n$ is the standard n-simplex with vertices $\{e_0, e_1, ..., e_n\}$. If J is a nonempty subset of $\{0, 1, ..., n\}$, then $\triangle_J$ stands for the face of $\triangle_n$ corresponding to J, i.e., $\triangle_J = co\{e_j : j \in J\}$.

Moreover, a set-valued map F is said to be intersectionally closed on X if

$$\bigcap_{x \in X} \mathrm{cl}F(x) = \mathrm{cl} \bigcap_{x \in X} F(x).$$

Recall that if X is a topological space and $F : X \to 2^Y$ a set-valued map, then F is upper semicontinuous on X if for each open set $V \subseteq Y$, the set $\{x \in X : F(x) \subseteq V\}$ is open in X.

2. Main results

In [6], by using set-valued maps, the authors introduced an abstract convex framework. Moreover, they proved a nonempty intersection result in such spaces.

Definition 2.1. Let X be a set, $(Y, \mathcal{U})$ be a uniform space and $\varphi : X \to 2^Y$ be a set-valued map. Then for a given $U \in \mathcal{U}$, φ is said to be small of order U if

$$\forall x \in X, \exists y \in Y \text{ s.t. } \varphi(x) \subseteq U(y).$$

It is clear that in the case where φ is a single-valued map, it is small of order U, for each $U \in \mathcal{U}$.

Recall that the concept of uniformity lies between semimetric and topological spaces, in the sense that every semimetric space is a uniform space, and every uniform space is a topological space. Compact Hausdorff spaces are uniformizable.

Definition 2.2. A triple (X, Y, ϕ) is called a uniform mapconvex space if X is a nonempty set, $(Y, \mathcal{U})$ is a uniform space and $\phi = \{\varphi_{N,U}\}_{(N,U)\in\langle X\rangle\times\mathcal{U}}$ is a collection of maps $\varphi_{N,U} : \triangle_N \to 2^Y$ which are upper semicontinuous with nonempty values and small of order U.

In the case where $X = Y$, we use the notation (X, ϕ) for uniform mapconvex space (X, X, ϕ).

It is easy to see that the convex subsets of topological vector spaces can be considered as uniform mapconvex spaces.

Definition 2.3. Let (X, Y, ϕ) be a uniform mapconvex space. A set-valued map $F : X \to 2^Y$ is said to be Φ-KKM if for each $(N, U) \in \langle X\rangle \times \mathcal{U}$ and for all $J \in \langle N\rangle$

$$\varphi_{N,U}(\triangle_J) \subseteq \bigcup_{x\in J} U(F(x)). \tag{2.1}$$

Theorem 2.4. *Let (X, Y, ϕ) be a uniform mapconvex space and Y be compact. Suppose that $F : X \to 2^Y$ is a Φ-KKM map. Then, the family of $\{\mathrm{cl}F(x) : x \in X\}$ has the finite intersection property. Furthermore, if F is intersectionally closed, then*

$$\bigcap_{x\in X} F(x) \neq \emptyset.$$

Here, by using Theorem 2.4, we present a new section theorem in Uniform mapconvex spaces.

Theorem 2.5. *Let (X, ϕ) be a compact Hausdorff Uniform mapconvex space and A be a nonempty subset of $X \times X$ such that $(x, x) \in A$ for all $x \in X$. Assume that*

(i) *For each $(N, U) \in \langle X\rangle \times \mathcal{U}$, $J \in \langle N\rangle$ and $x \in \varphi_{N,U}(\triangle_J)$ we have*

$$(J \times \{x\}) \cap A \neq \emptyset.$$

(ii) *For each $y \in X$, the set $\{x \in X : (x, y) \in A\}$ is closed in X.*

Then, there exists a point $x_0 \in X$, such that

$$\{x_0\} \times X \subset A.$$

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