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A NEW ACHIEVEMENT BETWEEN MULTIPLIER ALGEBRA AND OPERATOR ALGEBRA

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ABSTRACT. Let \mathcal{A} be a Banach algebra with a closed two sided ideal \mathcal{I} . We introduce \mathcal{I} -multipliers of \mathcal{A} , and prove that $\mathcal{M}_{\mathcal{I}}(\mathcal{A})$, the set of all \mathcal{I} -multipliers of \mathcal{A} is a closed subalgebra of $\mathcal{B}(\mathcal{A})$ containing $\mathcal{M}(\mathcal{A})$, where \mathcal{I} is an essential ideal of \mathcal{A} . Furthermore, we establish some basic properties of \mathcal{I} -multipliers.

1. INTRODUCTION

The concept of multipliers of a Banach algebra was introduced by Helgason [2] as follows:

Let \mathcal{A} be a semisimple Banach algebra considered as an algebra of continuous functions over its maximal ideal space $\Delta(\mathcal{A})$. Then a multiplier of \mathcal{A} is a function Φ over $\Delta(\mathcal{A})$ such that $\Phi\mathcal{A} \subseteq \mathcal{A}$. Every multiplier turns out to be a bounded continuous function and the set of all multipliers of \mathcal{A} under pointwise operations forms an algebra $\mathcal{M}(\mathcal{A})$, called the multiplier algebra of \mathcal{A} .

The general theory of multipliers on faithful Banach algebras has been developed by Wang in [5] and Birtal in [1].

In this paper, we generalize the Wang concept of multipliers to \mathcal{I} -multipliers, where \mathcal{I} is a closed two sided ideal of a Banach algebra

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\mathcal{A} and we prove that when \mathcal{I} is an essential ideal of \mathcal{A} , the set of all \mathcal{I} -multipliers of \mathcal{A} , $\mathcal{M}_{\mathcal{I}}(\mathcal{A})$, is a Banach algebra containing $\mathcal{M}(\mathcal{A})$. We present several examples of \mathcal{I} -multipliers which are not multipliers.

Finally, we extend some well-known results on multipliers to \mathcal{I} -multipliers.

2. PREMIMINARIES

A Banach algebra \mathcal{A} is said to be faithful if for every $a \in \mathcal{A}$, the condition $\mathcal{A}a = \{0\}$ implies $a = 0$. A mapping $T : \mathcal{A} \rightarrow \mathcal{A}$ is called a left (right) multiplier of \mathcal{A} if $T(xy) = x(Ty)$ ($(Tx)y$) holds for all $x, y \in \mathcal{A}$.

Let $\mathcal{M}^l(\mathcal{A})$ ($\mathcal{M}^r(\mathcal{A})$) denote the collection of all left (right) multipliers of \mathcal{A} and the set of all multipliers as follows:

$$\mathcal{M}(\mathcal{A}) = \mathcal{M}^l(\mathcal{A}) \cap \mathcal{M}^r(\mathcal{A}).$$

Denote by $\mathcal{B}(\mathcal{A})$ the Banach algebra of all continuous linear operators. For a faithful Banach algebra \mathcal{A} , the set $\mathcal{M}(\mathcal{A})$ of all multipliers of \mathcal{A} is a closed unital commutative subalgebra of $\mathcal{B}(\mathcal{A})$, called the *multiplier algebra* of \mathcal{A} [3].

It means that every multiplier in a faithful Banach algebra is linear and bounded. Examples of faithful Banach algebras are Banach algebras with approximate identity or commutative semisimple Banach algebras.

Suppose that \mathcal{A} is commutative and for $x \in \mathcal{A}$, define the multiplication operator $L_x : \mathcal{A} \rightarrow \mathcal{A}$ by $L_x y = xy$. Clearly $L_x \in \mathcal{M}(\mathcal{A})$ and the mapping $x \mapsto L_x$ is a continuous isomorphism of \mathcal{A} onto the ideal $\{L_x : x \in \mathcal{A}\}$ in $\mathcal{M}(\mathcal{A})$, which embeds \mathcal{A} in $\mathcal{M}(\mathcal{A})$. This leads us to the following inclusion:

$$\mathcal{A} \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{A}),$$

where \mathcal{A} is a faithful commutative Banach algebra.

If \mathcal{A} has an identity e , every multiplier T is of the form $T = L_{Te}$. It means that when \mathcal{A} has an identity, $\mathcal{M}(\mathcal{A}) = \mathcal{A}$. Now there is a natural question:

Question. When the equality $\mathcal{M}(\mathcal{A}) = \mathcal{B}(\mathcal{A})$ holds?

The general answer to this question is given by the following theorem, which we will apply in the sequel.

Theorem 2.1. *Let \mathcal{A} be a commutative faithful Banach algebra. The Banach algebra $\mathcal{M}(\mathcal{A})$ is equal to $\mathcal{B}(\mathcal{A})$ if and only if \mathcal{A} is one-dimensional.*

3. MAIN RESULTS

3.1. \mathcal{I} -multipliers of a Banach algebra. Let \mathcal{A} be a Banach algebra with a closed two sided ideal \mathcal{I} . A bounded linear operator on \mathcal{A} is called a left (right) \mathcal{I} -multiplier if it satisfies $T(xy) - xTy \in \mathcal{I}$ ($T(xy) - (Tx)y \in \mathcal{I}$), for all $x, y \in \mathcal{A}$. The set of all left (right) \mathcal{I} -multipliers of \mathcal{A} is denoted by $\mathcal{M}_{\mathcal{I}}^l(\mathcal{A})$ ($\mathcal{M}_{\mathcal{I}}^r(\mathcal{A})$) and we define the set of all \mathcal{I} -multipliers as

$$\mathcal{M}_{\mathcal{I}}(\mathcal{A}) = \mathcal{M}_{\mathcal{I}}^l(\mathcal{A}) \cap \mathcal{M}_{\mathcal{I}}^r(\mathcal{A}).$$

It is obvious that $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}_{\mathcal{I}}(\mathcal{A})$.

Here are some examples of \mathcal{I} -multipliers which are not multipliers.

Example 3.1. (1) Let X be a locally compact, but not compact, Hausdorff space and $\mathcal{A} = C_0(X)$. Let $x_0 \in X$ and define

$$\mathcal{I} = \mathcal{I}_{x_0} = \{f \in C_0(X) : f(x_0) = 0\}.$$

Obviously \mathcal{I}_{x_0} is a closed ideal in $C_0(X)$. Now let h be an arbitrary element of $C_0(X)$ and define:

$$\begin{aligned} T_h : C_0(X) &\rightarrow C_0(X) \\ T_h(f) &= f(x_0)h. \end{aligned}$$

It is evident that $T_h \in \mathcal{B}(\mathcal{A})$. For all $f, g \in C_0(X)$, we have

$$T_h(fg) - fT_h(g) = f(x_0)g(x_0)h - g(x_0)fh,$$

and we conclude that

$$(T_h(fg) - fT_h(g))(x_0) = 0,$$

meaning that $T_h(fg) - fT_h(g) \in \mathcal{I}$, for all $f, g \in C_0(X)$. It means that $T_h \in \mathcal{M}_{\mathcal{I}}(\mathcal{A})$. Now since X is a non compact space, for all $0 \neq f \in C_0(X)$, there is an $x_1 \in X$ such that $x_1 \neq x_0$ and $f(x_1) \neq f(x_0)$. Then $T_h(fg)(x_1) \neq fT_h(g)(x_1)$ and this implies that $T_h \notin \mathcal{M}(\mathcal{A})$. So that $\mathcal{M}(\mathcal{A})$ is a proper subset of $\mathcal{M}_{\mathcal{I}}(\mathcal{A})$.

(2) Again Let X be a locally compact, but not compact, Hausdorff space and $\mathcal{A} = C_0(X)$. Let $E = \{x_1, \dots, x_n\}$, $n \in \mathbb{N}$, be a finite subset of X and

$$\mathcal{I}_E = \{f \in C_0(X) : f(x) = 0, \text{ for all } x \in E\}.$$

Let $0 \neq h \in \mathcal{I}_E$ and define

$$T_h(f) = \left(\sum_{i=1}^n f(x_i) \right) h \quad (f \in C_0(X)).$$

Then for all $f, g \in C_0(X)$ and $x \in E$ we have

$$\begin{aligned} (T_h(fg) - fT_hg)(x) &= \\ \left(\sum_{i=1}^n fg(x_i) \right) h(x) - f(x) \left(\sum_{i=1}^n g(x_i) \right) h(x) &= 0, \end{aligned}$$

so that $(T_h(fg) - fT_hg) \in \mathcal{I}_E$, for all $f, g \in C_0(X)$ and consequently $T_h \in \mathcal{M}_{\mathcal{I}_E}(C_0(X))$. Now by Urysohn's Lemma there exists $f \in C_c(X) \subseteq C_0(X)$ such that $f|_E = 1$ so that $\sum_{i=1}^n f(x_i) = n \neq 0$. On the other hand, h is a nonzero element of $C_0(X)$, so there exists $y \in X$ such that $h(y) \neq 0$. Then we have

$$\begin{aligned} (T_h(hf) - hT_h(f))(y) &= \\ \sum_{i=1}^n h(x_i)f(x_i)h(y) - h(y) \sum_{i=1}^n f(x_i)h(y) &= \\ 0 - (h(y))^2 \sum_{i=1}^n f(x_i) &\neq 0. \end{aligned}$$

Hence $T_h \notin \mathcal{M}(C_0(X))$.

- (3) Let G be a locally compact group and \mathcal{I} be a closed two sided ideal in $\mathcal{A} = L^1(G)$. Let $0 \neq h \in \mathcal{I}$ and define $T_h \in \mathcal{B}(L^1(G))$ as follows

$$T_h(f) = \left(\int_G f(x) dx \right) h,$$

for every $f \in L^1(G)$. For any $f \in L^1(G)$, $T_h f \in \mathcal{I}$, meaning that $T_h \mathcal{A} \subseteq \mathcal{I}$ and $T \in \mathcal{M}_{\mathcal{I}}(\mathcal{A})$.

- (4) Let G be a locally compact abelian group and \widehat{G} the dual group of G . Consider an arbitrary subset $E \subseteq \widehat{G}$. Define

$$\mathcal{I}_E = \{f \in L^1(G) : \widehat{f}(\alpha) = 0, \text{ for all } \alpha \in E\}.$$

Then \mathcal{I}_E is a closed ideal of $L^1(G)$. Let $T \in \mathcal{B}(L^1(G))$ and $0 \neq h \in \mathcal{I}_E$. Define $S_h(f) := T(f) * h$. Then $S_h \mathcal{A} \subseteq \mathcal{I}$ and $S_h \in \mathcal{M}_{\mathcal{I}_E}(L^1(G))$.

In particular, let $G = \mathbb{R}$, the additive group of real numbers. It is well known that $\widehat{\mathbb{R}} = \mathbb{R}$ by the mapping $y \mapsto e^{ixy}$. If we consider the set of integers $\mathbb{Z} \subset \mathbb{R}$, then

$$\begin{aligned} \mathcal{I}_{\mathbb{Z}} = \{f \in L^1(\mathbb{R}) : \widehat{f}(x) = \int_{\mathbb{R}} f(x) e^{-ikx} dx = 0, \\ \text{for all } k \in \mathbb{Z}\} \end{aligned}$$

is a closed ideal of $L^1(\mathbb{R})$. Now define $h \in L^1(\mathbb{R})$ as follows:

$$h(x) = \begin{cases} e^{-ix} & -\pi \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Then $h \in \mathcal{I}_{\mathbb{Z}}$ and for an arbitrary operator $T \in \mathcal{B}(L^1(\mathbb{R}))$, the operator S_h defined by $S_h f = T f * h$, $f \in L^1(\mathbb{R})$, defines an $\mathcal{I}_{\mathbb{Z}}$ -multiplier of $L^1(\mathbb{R})$.

If we define S_h as

$$S_h f = \left(\int_{\mathbb{R}} f(x) dx \right) h,$$

for all $f \in L^1(\mathbb{R})$, then $S_h \in \mathcal{M}_{\mathcal{I}_{\mathbb{Z}}}(L^1(\mathbb{R}))$. If we consider $g = \chi_{(0,1)} \in L^1(\mathbb{R})$, the characteristic function of the open interval $(0,1)$, it is easily verified that

$$S_h(g * g) \neq g * S_h g.$$

So S_h is an $\mathcal{I}_{\mathbb{Z}}$ -multiplier of $L^1(\mathbb{R})$ that is not a multiplier.

Let \mathcal{A} be a Banach algebra and \mathcal{I} a closed ideal. We say that \mathcal{I} is an essential ideal in \mathcal{A} if

$$\mathcal{I} = \{ax : a \in \mathcal{A}, x \in \mathcal{I}\}.$$

Note that when \mathcal{A} is a Banach algebra with a bounded approximate identity, then by an application of the Cohen Factorization Theorem every closed ideal \mathcal{I} of \mathcal{A} is essential.

Proposition 3.2. *Let \mathcal{I} be an essential ideal of a commutative Banach algebra \mathcal{A} . Then*

$$\{T \in \mathcal{B}(\mathcal{A}) : T\mathcal{A} \subseteq \mathcal{I}\} \subseteq \mathcal{M}_{\mathcal{I}}(\mathcal{A}) \subseteq \{T \in \mathcal{B}(\mathcal{A}) : T\mathcal{I} \subseteq \mathcal{I}\}.$$

Here are two examples which show that the inclusions in Proposition 3.2 can be strict.

Example 3.3. (1) Let \mathcal{A} be a Banach algebra with a proper closed ideal \mathcal{I} and $T = id_{\mathcal{A}}$ be the identity operator. Then $T \in \mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}_{\mathcal{I}}(\mathcal{A})$, however $T\mathcal{A} = \mathcal{A} \not\subseteq \mathcal{I}$
 (2) Let \mathbb{R} be the additive group of real numbers and $\mathcal{A} = C_0(\mathbb{R})$. Consider the closed ideal

$$\mathcal{I} = \{f \in C_0(\mathbb{R}) : f(0) = f(1) = 0\}$$

of \mathcal{A} . Define $h \in C_0(\mathbb{R})$ as $h(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$ and $T \in \mathcal{B}(\mathcal{A})$ as follows:

$$Tf = (f(0) + f(1))h,$$

for all $f \in C_0(\mathbb{R})$. In this case, $T\mathcal{I} \subseteq \mathcal{I}$, however $T \notin \mathcal{M}_{\mathcal{I}}(\mathcal{A})$. Indeed, if we let $f = g = h \in C_0(X)$,

$$(T(fg) - fTg)(0) = -\frac{1}{4} \neq 0.$$

So $T(fg) - fTg \notin \mathcal{I}$.

Now we are ready to prove the following important properties of $\mathcal{M}_{\mathcal{I}}(\mathcal{A})$.

Theorem 3.4. *Let \mathcal{A} be a Banach algebra with an essential closed ideal \mathcal{I} . Then $\mathcal{M}_{\mathcal{I}}(\mathcal{A})$ is a closed unital subalgebra of $\mathcal{B}(\mathcal{A})$.*

Proposition 3.5. *Let \mathcal{A} be a commutative faithful Banach algebra with an essential closed ideal \mathcal{I} . Suppose that there exists a nonzero $x \in \mathcal{I}$ such that $xa \neq 0$ for all nonzero $a \in \mathcal{A}$. Then $\mathcal{M}_{\mathcal{I}}(\mathcal{A})$ is commutative if and only if \mathcal{A} is one-dimensional.*

Let \mathcal{A} be a Banach algebra and \mathcal{I} be a closed ideal in \mathcal{A} . If we consider the quotient Banach algebra $\frac{\mathcal{A}}{\mathcal{I}}$, and the quotient map

$$\pi : \mathcal{A} \rightarrow \frac{\mathcal{A}}{\mathcal{I}}, \quad \pi(x) = x + \mathcal{I},$$

then π is a surjective map and we can find a right inverse for π as follows:

Define

$$\rho : \frac{\mathcal{A}}{\mathcal{I}} \rightarrow \mathcal{A}, \quad \rho(x + \mathcal{I}) = C \cap \pi^{-1}(x + \mathcal{I}),$$

where $C \subseteq \mathcal{A}$ is the set that $\pi^{-1}(x + \mathcal{I}) \cap C$ contains just one point for every $x \in \mathcal{A}$ (This C exists by the Axiom of Choice).

Theorem 3.6. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity and \mathcal{I} be a closed ideal in \mathcal{A} . Let $T \in \mathcal{M}_{\mathcal{I}}(\mathcal{A})$. Then by the above notations we have*

$$\pi \circ T \circ \rho \in \mathcal{M}\left(\frac{\mathcal{A}}{\mathcal{I}}\right).$$

3.2. Some basic properties of $\mathcal{M}_{\mathcal{I}}(\mathcal{A})$. Let \mathcal{A} be a commutative Banach algebra and \mathcal{I} be a closed two sided ideal in Banach algebra \mathcal{A} . For a subset M of a commutative Banach algebra \mathcal{A} , the hull $h(M)$ of M is defined to be

$$h(M) = \{\varphi \in \Delta(\mathcal{A}) : \varphi(M) = \{0\}\}.$$

Let \mathcal{I} be a closed ideal of \mathcal{A} and $q : \mathcal{A} \rightarrow \frac{\mathcal{A}}{\mathcal{I}}$ the quotient homomorphism. Then the map $\varphi \mapsto \varphi o q$ is a homeomorphism from $\Delta(\frac{\mathcal{A}}{\mathcal{I}})$ onto $h(\mathcal{I})$. See Lemma 2.2.15 of [4].

Theorem 3.7. *Let \mathcal{A} be a commutative Banach algebra with a closed ideal \mathcal{I} and let $T \in \mathcal{M}_{\mathcal{I}}(\mathcal{A})$. Then there exists a unique continuous function Φ on $h(\mathcal{I})$ such that $\widehat{Tx}(\varphi) = \Phi(\varphi)\widehat{x}(\varphi)$ for all $\varphi \in h(\mathcal{I})$ and $x \in \mathcal{A}$. Furthermore Φ is bounded and $\|\Phi\|_{\infty} \leq \|T\|$.*

For the next characterization of $\mathcal{M}_{\mathcal{I}}(\mathcal{A})$, we first need to prove the following lemma:

Lemma 3.8. *Let \mathcal{A} be a Banach algebra with a closed two sided ideal \mathcal{I} such that the quotient algebra $\frac{\mathcal{A}}{\mathcal{I}}$ is faithful. Then for any $T, S \in \mathcal{M}_{\mathcal{I}}(\mathcal{A})$, we have*

$$TS(x) - ST(x) \in \mathcal{I} \quad (x \in \mathcal{A}).$$

Now we turn to the following result.

Theorem 3.9. *Let \mathcal{A} be a commutative Banach algebra with a closed ideal \mathcal{I} for which the quotient algebra $\frac{\mathcal{A}}{\mathcal{I}}$ is faithful and T a linear mapping from \mathcal{A} to \mathcal{A} . Then the following statements are equivalent:*

- (1) $T \in \mathcal{M}_{\mathcal{I}}(\mathcal{A})$.
- (2) $Tx^2 - x(Tx) \in \mathcal{I}$ for all $x \in \mathcal{A}$.

Theorem 3.10. *Let \mathcal{A} be a Banach algebra with a closed two sided essential ideal \mathcal{I} and $T \in \mathcal{M}_{\mathcal{I}}(\mathcal{A})$. Then the following are equivalent:*

- (1) T is bijective and $T\mathcal{I} = \mathcal{I}$.
- (2) T^{-1} exists and $T^{-1} \in \mathcal{M}_{\mathcal{I}}(\mathcal{A})$.

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