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ON CYCLIC STRONGLY QUASI-CONTRACTION MAPS

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Abstract. Let A and B be nonempty subsets of a metric space (X, d) and self mapping $T : A \cup B \to A \cup B$ be a cyclic map. In 2013 Amini-Harandi ['Best proximity point theorems for cyclic strongly quasi-contraction mappings', J. Global Optim. 56 (2013), 1667-1674] introduced the notion of maps called cyclic strongly quasi-contraction, with adding the condition

$$d(T^2x, T^2y) \leq c\ d(x, y) + (1 - c)d(A, B)$$
$$for\ all\ x \in A\ and\ y \in B\ where\ c \in [0, 1), \tag{I}$$

to cyclic quasi-contraction maps and proved an existence result of best proximity point theorem. The author also posed the question that does this theorem remains true for cyclic quasi-contraction maps? In 2017, Dung and Hang gave negative answer to question of Amini-Harandi and decided to prove his theorem. But they had mistakes in proving theorem. In this paper, first we show that the condition (I) is so strong that theorem of Amini-Harandi (and so modified version of it) is correct by using it alone.

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1. Introduction

Let A and B be nonempty subsets of a metric space (X, d). A self mapping $T : A \cup B \to A \cup B$ is called noncyclic provided that $T(A) \subseteq A$ and $T(B) \subseteq B$, and is said to be cyclic provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x \in A \cup B$ is called a best proximity point for T if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$. In 2013 Amini-Harandi [1] introduced a new class of maps called cyclic strongly quasi-contractions, as following.

Definition 1.1. [1] Let A and B be nonempty subsets of a complete metric space (X, d) and let T be a cyclic mapping on $A \cup B$. The map T is said to be cyclic quasi-contraction if

$$\begin{aligned} d(Tx, Ty) \leq c \max\big\{&d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\} \\ &+ (1 - c)d(A, B), \end{aligned} \tag{1.1}$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$, and is said to be cyclic strongly quasi-contraction if in addition to the condition (1.1) we have

$$d(T^2x, T^2y) \leq c\ d(x, y) + (1 - c)d(A, B), \tag{1.2}$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$.

The main result of [1] is as follows.

Theorem 1.2. [1] *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let T be a cyclic strongly quasi-contraction mapping on $A \cup B$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then there exists a unique $x^* \in A$ such that $\{x_{2n}\}$ is converges to x^*, $T^2x^* = x^*$ and $\|x^* - Tx^*\| = d(A, B)$.*

The author also mentioned the following question.

Question 1.3. *Dose the conclusion of Theorem 1.2 remains true for cyclic quasi-contraction maps?*

In 2016 Dung, Radenovic [4] proved following theorem.

Theorem 1.4. [4] *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let T be a cyclic mapping on $A \cup B$ such that for all $x \in A$ and $y \in B$ and some $c \in [0, 1)$ the conditions (1.2) and*

$$\begin{aligned} d(Tx, Ty) \leq &c\ \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \\ &+ (1 - c)d(A, B), \end{aligned} \tag{1.3}$$

hold. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then there exists a unique $x^* \in A$ such that $\{x_{2n}\}$ is converges to x^*, $T^2x^* = x^*$ and $\|x^* - Tx^*\| = d(A, B)$.

In 2017 Dung, Hang [3] gave negative answer to Question 1.3 and decided to prove Theorem 1.2. Unexpectedly, in the proof, the authors used the cyclic quasi-contraction condition (1.1) in the last page, for the pair (x, x_{2n}) which belong to $A \times A$. This is inappropriate since the cyclic quasi-contraction condition (1.1) only holds for pairs in $A \times B$. In this paper, we show that the quasi-contrction condition (1.2) is so strong that Theorem 1.2 (resp.1.4) is correct by using it alone. In fact, it is obtained as a result of a fixed point theorem in [7]. Also, we prove that the condition (1.3) is not sufficient to establish Theorem 1.4, actually we show that Theorem 4.5 [5] can not be true, generally. In the end of paper, we obtain a fixed point theorem.
In the following we give some basic definitions and concepts which are useful and related to the context of our results. Let A and B be nonempty subsets of a metric space (X, d) and let T be a cyclic (resp. noncyclic) mapping on $A \cup B$. Then, T is said to be a cyclic (resp. noncyclic) contraction map if

$$d(Tx, Ty) \leq c\, d(x, y) + (1 - c)d(A, B),$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$.

Lemma 1.5. [2] *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space X. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying*

(i) $\|x_n - y_n\| \to d(A, B)$.
(ii) $\|z_n - y_n\| \to d(A, B)$.

Then $\|x_n - z_n\|$ converges to zero.

Theorem 1.6. [7] *Let A and B be two closed convex subsets of a strictly convex and reflexive Banach space X. Suppose that T is a noncyclic contraction map on $A \cup B$. Then T has a best proximity pair, that is there exist fixed points $x^* \in A$ and $y^* \in B$ such that $d(x^*, y^*) = d(A, B)$.*

2. Main results

We begin this section with a simple but useful lemma.

Lemma 2.1. *Let A and B be nonempty subsets of a metric space (X, d). Suppose that T is a noncyclic contraction map on $A \cup B$. For*

$x_0 \in A$ and $y_0 \in B$, define $x_{n+1} := Tx_n$ and $y_{n+1} := Ty_n$ for each $n \geq 0$. Then

$$\lim_{n\to\infty} d(x_n, y_n) = d(A, B).$$

Corollary 2.2. *Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach space X. Suppose that T is a noncyclic contraction map on $A \cup B$. For $x_0 \in A$ and $y_0 \in B$, define $x_{n+1} := Tx_n$ and $y_{n+1} := Ty_n$ for each $n \geq 0$. Then T has a unique best proximity pair (x^*, y^*) such that $\{x_n\}$ is converges to x^* for every $x_0 \in A$ and $\{y_n\}$ is converges to y^* for every $y_0 \in B$.*

Theorem 2.3. *Theorem 1.2 (resp. 1.4) without the condition (1.1) (resp. (1.3)) is a consequence of Theorem 1.6.*

According to the above discussion, the definition of cyclic strongly quasi-contraction mappings is unnecessary and inappropriate.
We are now ready to discuss in Question 1.3. In 2017 Dung, Hang [3] gave negative answer to this question in the case $d(A, B) = 0$. In the following we give a negative answer to this question in the case $d(A, B) \neq 0$, too. We show that the conclusions of Theorem 1.2 are not hold for cyclic quasi-contraction maps, in the case $d(A, B) \neq 0$.

Example 2.4. Let $X = \mathbb{R}^3$ with the Euclidean norm, $a = (0, 0, 1), a' = (2, 2, 1), a'' = (1, 1, 1), b = (0, 2, 0), b' = (2, 0, 0), b'' = (1, 1, 0)$, A be the segment with two endpoints a, a' and B be the segment with two endpoints b, b' and

$$Ta = b,\ Ta' = b',\ Tb = a',\ Tb' = a,\ Tx = b''\ for\ x \in A \setminus \{a, a'\}$$
$$,\ Ty = a''\ for\ y \in B \setminus \{b, b'\}.$$

Then X is a uniformly convex Banach space and A and B are nonempty closed convex sets in X. T is cyclic on $A \cup B$ and $d(A, B) = 1$. We will check that T satisfies (1.1) by exhausting the following cases.

Case 1. $x = a$, $y = b$. Then $d(Tx, Ty) = d(b, a') = \sqrt{5}$ and $d(x, Ty) = d(a, a') = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(x, Ty)$.

Case 2. $x = a$, $y = b'$. Then $d(Tx, Ty) = d(b, a) = \sqrt{5}$ and $d(y, Tx) = d(b', b) = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(y, Tx)$.

Case 3. $x = a$, $y \in B \setminus \{b, b'\}$. Then $d(Tx, Ty) = d(b, a'') = \sqrt{3}$ and $d(x, Tx) = d(a, b) = \sqrt{5}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(x, Tx)$.

Case 4. $x = a'$, $y = b$. Then $d(Tx, Ty) = d(b', a') = \sqrt{5}$ and $d(y, Tx) = d(b, b') = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(y, Tx)$.

Case 5. $x = a'$, $y = b'$,. Then $d(Tx, Ty) = d(b', a) = \sqrt{5}$ and $d(x, Ty) = d(a', a) = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\ d(x, Ty)$.
Case 6. $x = a'$, $y \in B \setminus \{b, b'\}$. Then $d(Tx, Ty) = d(b', a'') = \sqrt{3}$ and $d(x, Tx) = d(a', b') = \sqrt{5}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\ d(x, Tx)$.
Case 7. $x \in A \setminus \{a, a'\}$, $y = b$. Then $d(Tx, Ty) = d(b'', a') = \sqrt{3}$ and $d(y, Ty) = d(b, a') = \sqrt{5}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\ d(y, Ty)$.
Case 8. $x \in A \setminus \{a, a'\}$, $y = b'$. Then $d(Tx, Ty) = d(b'', a) = \sqrt{3}$ and $d(y, Ty) = d(b', a) = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\ d(y, Ty)$.
Case 9. $x \in A \setminus \{a, a'\}$, $y \in B \setminus \{b, b'\}$. Then $d(Tx, Ty) = d(b'', a'') = 1$ and $d(x, y) > 1$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\ d(x, y) + (1 - \sqrt{\frac{5}{8}})d(A, B)$.

By the above nine cases, we have

$$d(Tx, Ty) \leq \sqrt{\frac{5}{8}} \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\} + (1 - \sqrt{\frac{5}{8}})d(A, B).$$

So T is a cyclic quasi-contraction map, but for $x_0 = a \in A$, the sequence $\{x_{2n}\}$ is not convergent, where $x_{n+1} = Tx_n$ for each $n \geq 0$. □

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