The Extended Abstracts of
The 6^{th} Seminar on Functional Analysis and its Applications
4-5th March 2020, University of Isfahan, Iran

A SHEARLET APPROACH TO NUMERICAL SOLUTION OF N-DIMENSIONAL HEAT EQUATION

R. AMIRI *, [1] M. ZAREBNIA [2] R. RAISI TOUSI [3]

[1] *Department of Mathematics and Applications, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran.*
r.amiri@uma.ac.ir
[2] *Department of Mathematics and Applications, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran.*
Zarebnia@uma.ac.ir
[3] *Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.*
raisi@um.ac.ir

Abstract. Shearlet frames are used to solve N-dimensional heat equation numerically. The proposed method is mathematically simple and fast. To demonstrate the performance and efficiency of the developed formulation an example in the case of two-dimensions is presented. By this approach the coefficients of the shearlet frame expansion are obtained via separate time independent partial differential equations and the number of these coefficients are decided according to the desirable accuracy of the solution.

1. Introduction

Heat equation is categorized as parabolic second order partial differential equations. This kind of equations appear in different scientific fields. Several numerical techniques have so far been developed for solution of transient heat transfer problems. Among these methods,

2010 *Mathematics Subject Classification.* Primary: 42C15; Secondary:35K05 .
Key words and phrases. Shearlets;Heat equation.
* Speaker.

finite difference methods [16, 14, 17], finite element methods [15, 20], spectral methods [5, 19], etc. can be mentioned.

Amongst the numerical approaches the wavelet-based numerical methods have been developed and have widely used to solve the various types of heat equation in different dimensions, for examples see [2, 4, 9, 10]. Curvelets have also been employed for solution of PDE problems, e.g. solution of two dimensional wave equation [18].

Shearlets are newer representation systems that are equipped with a rich mathematical structure similar to wavelets [6, 12, 13]. In fact, theory and algorithms of shearlets can be carried over the continuous wavelet transform. The continuous shearlet transform is based on special affine systems generated by one single function $\psi \in L^2(\mathbb{R}^2)$. Moreover, compared with wavelets, the continuous shearlet transform has a coherent matrix structure for n-dimensions so that it is useful for solving the higher dimensional PDEs [1].

In this paper, an approach for solution of heat equation in the general case of n-dimensions is presented. The unknown function is expanded by using shearlet frames and then by employing Fourier transform and making use of Plancherel theorem, as well as properties of shearlets, the unknown coefficients of the expansion are obtained by solving far simpler separate time independent partial differential equations.

The paper is organized as follows. In the rest of this section, we present some necessary definitions and theorems. Section 2 is devoted to the development of n-dimensional formulation. In section 3 an example of a two-dimensional heat problem is presented. The conclusions and merits of the approach are concisely discussed in section 4.

Firstly, we present required notation and definitions about shearlets.

Let $\{\psi_{j,k,m}(\cdot)\}_{j,k,m}$ be a family of shearlets in n-dimensions as

$$\psi_{j,k,m}(\cdot) = |\det A_{2^j}|^{-\frac{1}{2}} \psi(A_{2^j}^{-1} S_k^{-1}(\cdot - m)), \tag{1.1}$$

where $j \in \mathbb{Z}, k \in \mathbb{Z}^{n-1}, m \in \mathbb{Z}^n$ and $\psi \in L^2(\mathbb{R}^n)$ is admissible in the sense that

$$C_\psi = \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\xi)|^2}{|\xi_1|^2} d\xi_n \cdots d\xi_1 < \infty, \tag{1.2}$$

and

$$A_{2^j} = \begin{bmatrix} 2^j & 0_{n-1}^T \\ 0_{n-1} & 2^{\frac{j}{2}} I_{n-1} \end{bmatrix}, \quad S_k = \begin{bmatrix} 1 & k^T \\ 0_{n-1} & I_{n-1} \end{bmatrix}, \tag{1.3}$$

in special case, $n = 2$, we have

$$\psi_{j,k,m}(\cdot) = 2^{-\frac{3}{4}} \psi(A_{2^j}^{-1} S_k^{-1}(\cdot - m)),$$

where $j, k \in \mathbb{Z}, m \in \mathbb{Z}^2$,

$$A_{2^j} = \begin{bmatrix} 2^j & 0 \\ 0 & 2^{\frac{j}{2}} \end{bmatrix}, \ S_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix},$$

Definition 1.1. Let $\psi_1 \in L^2(\mathbb{R})$ be a admissible wavelet with $\widehat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\widehat{\psi}_1 \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. Consider $\psi_2 \in L^2(\mathbb{R}^{n-1})$ be such that $\widehat{\psi}_2 \in C^\infty(\mathbb{R}^{n-1})$ and supp $\widehat{\psi}_2 \subseteq [-1, 1]^{n-1}$, then the function $\psi \in L^2(\mathbb{R}^n)$ defined by

$$\widehat{\psi}(\xi) = \widehat{\psi}(\xi_1, \tilde{\xi}) = \widehat{\psi}_1(\xi_1) \cdot \widehat{\psi}_2(\frac{\tilde{\xi}}{\xi_1}),$$

where $\tilde{\xi} = (\xi_2, \cdots, \xi_n)$ is a continuous shearlet.

Let $\psi_1 \in L^2(\mathbb{R})$ satisfies the discrete Calderon's condition

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi}_1(2^{-j}\xi)|^2 = 1,$$

with $\widehat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\widehat{\psi}_1 \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. Consider $\psi_2 \in L^2(\mathbb{R}^n)$ is a bump function such that for all $\xi \in [-1, 1]^{n-1}$,

$$\sum_{k=-1}^{1} |\widehat{\psi}_2(\xi + k)|^2 = 1,$$

where $\widehat{\psi}_2 \in C^\infty(\mathbb{R}^{n-1})$ and supp $\widehat{\psi}_2 \subseteq [-1, 1]^{n-1}$.

For instance, for $n = 2$. Let $\psi_1 \in L^2(\mathbb{R})$ be a Lemarie'-Meyer wavelet that satisfies the discrete Calderon's condition

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi}_1(2^{-j}\xi)|^2 = 1,$$

with $\widehat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\widehat{\psi}_1 \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. Consider $\psi_2 \in L^2(\mathbb{R})$ is a bump function such that for all $\xi \in [-1, 1]$,

$$\sum_{k=-1}^{1} |\widehat{\psi}_2(\xi + k)|^2 = 1,$$

where $\widehat{\psi}_2 \in C^\infty(\mathbb{R})$ and supp $\widehat{\psi}_2 \subseteq [-1, 1]$.

We have ψ as

$$\widehat{\psi}(\xi_1, \xi_2) = \widehat{\psi}_1(\xi_1)\widehat{\psi}_2(\frac{\xi_2}{\xi_1}).$$

The shearlet ψ as defined in above is called a classical shearlet [12].

For case of n, we will show that the classical shearlet can be defined as follows

$$\hat{\psi}(\xi_1, \tilde{\xi}) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\tilde{\xi}}{\xi_1}),$$

where $\hat{\psi}_1, v(x), b(\xi)$ is the same of example in [3] and consider $I = \{i_1, i_2, \cdots, i_{n_i}\} \subseteq \{2, 3, \cdots, n\}$; $J = \{j_1, j_2, \cdots, j_{n_j}\} \subseteq \{2, 3, \cdots, n\}$ be such that $I \cap J = \emptyset$. Now, we defined $\hat{\psi}_2(\tilde{\xi})$ as follows

$$\hat{\psi}_2^{\,2}(\tilde{\xi}) = \hat{\psi}_2^{\,2}(\xi_2, \xi_3, \cdots, \xi_n) = v(1-\xi_{i_1}) \cdots v(1-\xi_{i_{n_i}})v(1+\xi_{j_1}) \cdots v(1+\xi_{j_{n_j}})$$

where v is the function defined in [3],
$\xi_{i_{n'}} \geq 0$ for $n' = 1, 2, \cdots, n_i$,
$\xi_{i_{n''}} \leq 0$ for $n'' = 1, 2, \cdots, n_j$,
and $n_i + n_j = n - 1$ It can be shown that $\hat{\psi}_2$ are satisfied in the conditions mentioned in definition 1.1.

In the following, we will show that $\hat{\psi}_2$ satisfies the conditions in Definition 1.1. Consider $\tilde{k} = (k_2, k_3, \cdots, k_n)$ where $k_\alpha \in \mathbb{Z}$, $\alpha \in \{2, 3, \cdots, n\}$, specially we could consider $k_\alpha \in [-2^j, 2^j]$ where j is the scale parameter. In the following we show that

$$\sum_{k_2=-2^j}^{2^j} \sum_{k_3=-2^j}^{2^j} \cdots \sum_{k_n=-2^j}^{2^j} \hat{\psi}_2(\tilde{\xi} + \tilde{k})|^2 = 1. \tag{1.4}$$

By definition of $\hat{\psi}_2(\tilde{\xi})$ we can write

$$\begin{aligned}
&\sum_{k_2} \sum_{k_3} \cdots \sum_{k_n} |\hat{\psi}_2(\tilde{\xi} + \tilde{k})|^2 \\
&= \sum_{k_{i_1}} \sum_{k_{i_2}} \cdots \sum_{k_{n_j}} v(1 - \xi_{i_1} + k_{i_1}) \cdots v(1 - \xi_{i_{n_i}} + k_{i_{n_i}}) v(1 + \xi_{j_1} + k_{j_1}) \cdots \\
&\qquad\qquad v(1 + \xi_{j_{n_j}} + k_{j_{n_j}}) \\
&= \sum_{k_{i_1}} v(1 - \xi_{i_1} + k_{i_1}) \cdots \sum_{k_{i_{n_i}}} v(1 - \xi_{i_{n_i}} + k_{i_{n_i}}) \sum_{k_{j_1}} v(1 + \xi_{j_1} + k_{j_1}) \cdots \\
&\qquad\qquad \sum_{k_{j_{n_j}}} v(1 + \xi_{j_{n_j}} + k_{j_{n_j}}) \\
&= 1,
\end{aligned}$$

(see Theorem 2.5 in [3]) Moreover by (1.4) and Theorem 2.2 in [3] it can be easily seen that

$$\sum_{j,k} |\hat{\psi}_{j,k}(\xi)|^2 = 1.$$

where $\xi = (\xi_1, \cdots, \xi_n),\ k = (k_1, \cdots, k_n)$.

Now, we can proof the following theorem.

Theorem 1.2. *The shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ in (1.1) with Definition 1.1 is a Parseval frame for $L^2(\mathbb{R}^n)$.*

Proof. Suppose $f \in L^2(\mathbb{R}^n)$ and $\hat{f}$ is Fourier transform of f, then we have

$$\begin{aligned} \sum_{j,k,m} |\langle f, \psi_{j,k,m}\rangle|^2 &= \sum_{j,k,m} |\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle|^2 \\ &= \sum_{j,k,m} |\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{\psi}}_{j,k,m}(\xi) d\xi|^2 \\ &= \sum_{j,k,m} |\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{\psi}}_{j,k}(\xi) e^{i\langle m,\xi\rangle} d\xi|^2 \\ &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{j,k} |\hat{\psi}_{j,k}(\xi)|^2 d\xi, \end{aligned}$$

since $\sum_{j,k} |\hat{\psi}_{j,k}(\xi)|^2 = 1$, as show in (1.4) , so

$$\sum_{j,k,m} |\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle|^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |f(x)|^2 dx = \|f\|^2.$$

□

Now, we define $\tilde{\psi}_{j,k,m}$ as follows [8]

$$\tilde{\psi}_{j,k,m} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$$

$$\hat{\tilde{\psi}}_{j,k,m}(\xi, t) = \overline{\hat{\psi}}_{j,k,m}(\xi) e^{\pm i|\xi|ct}, \tag{1.5}$$

where ψ is a classical shearlet. In the following, we represent $\{\tilde{\psi}_{j,k,m}\}_{j,k,m}$ be a Parseval frame.

Theorem 1.3. *Let $\psi \in L^2(\mathbb{R}^n)$ be a classical shearlet. The shearlet system $\{\tilde{\psi}_{j,k,m}\}_{j,k,m}$ is a Parseval frame.*

Proof. Suppose $f \in L^2(\mathbb{R}^n)$, $\hat{f}$ is Fourier transform of f and $\hat{f}(\xi,t) = \hat{f}(\xi)e^{\pm i|\xi|ct}$, then we have

$$\begin{aligned}
\sum_{j,k,m} |\langle f, \tilde{\psi}_{j,k,m}\rangle|^2 &= \sum_{j,k,m} |\langle \hat{f}, \hat{\tilde{\psi}}_{j,k,m}\rangle|^2 \\
&= \sum_{j,k,m} |\int_{\mathbb{R}^n} \hat{f}(\xi)e^{\pm i|\xi|ct}\overline{\hat{\psi}}_{j,k,m}(\xi)e^{\mp i|\xi|ct}d\xi|^2 \\
&= \sum_{j,k,m} |\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{\psi}}_{j,k}(\xi)e^{i\langle m,\xi\rangle}d\xi|^2 \\
&= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{j,k} |\hat{\psi}_{j,k}(\xi)|^2 d\xi,
\end{aligned}$$

by [3], since $\sum_{j,k} |\hat{\psi}_{j,k}(\xi)|^2 = 1$, then

$$\sum_{j,k,m} |\langle f, \tilde{\psi}_{j,k,m}\rangle|^2 = \|f\|^2.$$

□

Since $\{\tilde{\psi}_{j,k,m}\}_{j,k,m}$ is a Parseval frame for $L^2(\mathbb{R}^n)$, then we can write $f \in L^2(\mathbb{R}^n)$ as follows

$$f(x,t) = \sum_{j,k,m} \langle f, \tilde{\psi}_{j,k,m}\rangle \tilde{\psi}_{j,k,m}(x,t). \tag{1.6}$$

We use $C_{j,k,m}$ to show the shearlet coefficients $\langle f, \tilde{\psi}_{j,k,m}\rangle$. In the next section, we present a method to solving the n-dimensional heat equation with the shearlet frame (1.5).

2. Main results

In this section, we present a method for solving n-dimensional heat equation using shearlet frame. First, we consider n-dimensional heat equation as

$$\begin{cases}
u_t = c'^2 \Delta u, \qquad 0 \le x_i \le a_i, \ i = 1, \cdots, n \\
u(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n, t) = u(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_n, t) = 0, \\
\qquad\qquad 0 \le x_j \le a_j, j \ne i, \ t \ge 0,
\end{cases} \tag{2.1}$$

where $\Delta u = \sum_{i=1}^{n} (\frac{\partial^2 u}{\partial x_i^2})$.

Applying n-dimensional Fourier transform with respect to x yields

$$\frac{\partial}{\partial t}\widehat{U}(\xi,t) = -(\xi_1^2 + \cdots + \xi_n^2)\widehat{U}(\xi,t). \tag{2.2}$$

Then we consider

$$u(x,t) = \sum_{j,k,m} C_{j,k,m}\tilde{\psi}_{j,k,m}(x,t), \tag{2.3}$$

and

$$\Delta u(x,t) = \sum_{j,k,m} C^{\Delta}_{j,k,m}\tilde{\psi}_{j,k,m}(x,t),$$

where

$$\begin{aligned} C_{j,k,m} &= \langle u(x,t), \tilde{\psi}_{j,k,m}(x,t)\rangle \\ C^{\Delta}_{j,k,m} &= \langle \Delta u(x,t), \tilde{\psi}_{j,k,m}(x,t)\rangle. \end{aligned} \tag{2.4}$$

Applying n-D Fourier transform to (2.4) and using Plancherel theorem yields

$$\begin{aligned} \widehat{U}(\xi,t) &= \sum_{j,k,m} C_{j,k,m}\hat{\tilde{\psi}}_{j,k,m}(\xi,t) \\ \widehat{\Delta U}(\xi,t) &= \sum_{j,k,m} C^{\Delta}_{j,k,m}\hat{\tilde{\psi}}_{j,k,m}(\xi,t). \end{aligned} \tag{2.5}$$

By replacing $\widehat{U}$ and $\widehat{\Delta U}$ into (2.2), we obtain

$$\sum_{j,k,m} C_{j,k,m}\frac{\partial}{\partial t}\hat{\tilde{\psi}}_{j,k,m}(\xi,t) = c'^2 \sum_{j,k,m} C^{\Delta}_{j,k,m}\hat{\tilde{\psi}}_{j,k,m}(\xi,t). \tag{2.6}$$

$$\sum_{j,k,m} C_{j,k,m}(\pm i|\xi|c)\hat{\psi}_{j,k,m}(\xi)e^{\mp i|\xi|ct} = c'^2 \sum_{j,k,m} C^{\Delta}_{j,k,m}\hat{\psi}_{j,k,m}(\xi)e^{\mp i|\xi|ct}. \tag{2.7}$$

Now, (2.7) can be rewritten as follows

$$\sum_{j,k,m} \left[\pm i|\xi|C_{j,k,m} - C^{\Delta}_{j,k,m}\right]\hat{\psi}_{j,k,m} = 0, \tag{2.8}$$

$$\pm i|\xi|C_{j,k,m} - C^{\Delta}_{j,k,m} = 0. \tag{2.9}$$

By using the definition of $C_{j,k,m}, C^{\Delta}_{j,k,m}$ and the Plancherel theorem, we get

$$C_{j,k,m} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{U}(\xi,t).\overline{\hat{\hat{\psi}}_{j,k,m}(\xi,t)}d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{U}(\xi).\overline{\hat{\psi}_{j,k,m}(\xi)}d\xi,$$

$$\begin{aligned}C^{\Delta}_{j,k,m} &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{\Delta U}.\overline{\hat{\hat{\psi}}}_{j,k,m}(\xi,t)d\xi \\ &= -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\xi_1^2+\cdots+\xi_n^2)\widehat{U}(\xi).\overline{\hat{\psi}_{j,k,m}(\xi)}d\xi \\ &= -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\xi_1^2+\cdots+\xi_n^2)\widehat{U}(\xi).\overline{\hat{\psi}_{j,k}(\xi)}e^{i\langle m,\xi\rangle}d\xi.\end{aligned} \tag{2.10}$$

Substituting $S^T_{-k}A_{2^{-j}}\xi$ with ξ, we have

$$C_{j,k,m} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} A_{2^j}S^T_k\left(\widehat{U}(\xi).\overline{\hat{\psi}}_{j,k}(\xi)\right)e^{i\langle m,\xi\rangle}d\xi,$$

$$C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |A_{2^j}S^T_k\xi|^2 A_{2^j}S^T_k\left(\widehat{U}(\xi).\overline{\hat{\psi}}_{j,k}(\xi)\right)e^{i\langle m,\xi\rangle}d\xi.$$

For simplicity, we consider $\Gamma := A_{2^j}S^T_k\left(\widehat{U}(\xi).\overline{\hat{\psi}}_{j,k}(\xi)\right)$, hence, $C_{j,k,m}$ and $C^{\Delta}_{j,k,m}$ can be rewritten as

$$C_{j,k,m} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \Gamma e^{i\langle m,\xi\rangle}d\xi, \tag{2.11}$$

and

$$C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |A_{2^j}S^T_k\xi|^2\Gamma e^{i\langle m,\xi\rangle}d\xi. \tag{2.12}$$

Expanding $|A_{2^j}S_k^T\xi|^2$ in the above relation yields

$$\begin{aligned}
C^{\Delta}_{j,k,m} = 2^j[&(2^j+k^2(n-1))\xi_1^2+(1+k^2(n-2))\xi_2^2+(1+k^2(n-3))\xi_3^2\\
&+\cdots+(1+k^2)\xi_{n-1}^2+\xi_n^2\\
&+(2k+2k^2(n-2))\xi_1\xi_2+(2k+2k^2(n-3))\xi_1\xi_3\\
&+(2k+2k^2(n-4))\xi_1\xi_4+\cdots+2k\xi_1\xi_n\\
&+(2k+2k^2(n-3))\xi_2\xi_3+(2k+2k^2(n-4))\xi_2\xi_4\\
&+\cdots+(2k)\xi_2\xi_n+\\
&\vdots\\
&+(2k)\xi_{n-1}\xi_n]=\Theta(\xi).
\end{aligned} \tag{2.13}$$

So $C^{\Delta}_{j,k,m}$ can be rewritten as follows

$$C^{\Delta}_{j,k,m}=-\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}[\Theta(\xi)]\Gamma e^{i\langle m,\xi\rangle}d\xi,$$

it could be seen that $C^{\Delta}_{j,k,m}$ is a combination of following terms

$$C^{\Delta}_{k_1} = (2^j + k^2(n-1))\tfrac{\partial^2 C_{j,k,m}}{\partial m_1^2} + (1 + k^2(n-2))\tfrac{\partial^2 C_{j,k,m}}{\partial m_2^2}$$

$$+ \cdots + (1+k^2)\tfrac{\partial^2 C_{j,k,m}}{\partial m_{n-1}^2} + \tfrac{\partial^2 C_{j,k,m}}{\partial m_n^2}$$

$$C^{\Delta}_{k_1 k_n} = (2k + 2k^2(n-2))\tfrac{\partial^2 C_{j,k,m}}{\partial m_1 \partial m_2} + (2k + 2k^2(n-3))\tfrac{\partial^2 C_{j,k,m}}{\partial m_1 \partial m_3}$$

$$+ \cdots + 2k\tfrac{\partial^2 C_{j,k,m}}{\partial m_1 \partial m_n}$$

$$\vdots$$

$$C^{\Delta}_{k_2 k_n} = (2k + 2k^2(n-3))\tfrac{\partial^2 C_{j,k,m}}{\partial m_2 \partial m_3} + (2k + 2k^2(n-4))\tfrac{\partial^2 C_{j,k,m}}{\partial m_2 \partial m_4}$$

$$+ \cdots + (2k)\tfrac{\partial^2 C_{j,k,m}}{\partial m_2 \partial m_n}$$

$$\vdots$$

$$C^{\Delta}_{k_{n-1} k_n} = (2k)\tfrac{\partial^2 C_{j,k,m}}{\partial m_{n-1} \partial m_n}. \tag{2.14}$$

Replacing (2.14) in (2.9) leads to

$$\mp i|\xi| C_{j,k,m} - {c'}^2 \left[C^{\Delta}_{k_1} + C^{\Delta}_{k_1 k_n} + C^{\Delta}_{k_2 k_n} + \cdots + C^{\Delta}_{k_{n-1} k_n} \right] = 0. \tag{2.15}$$

Where (2.15) is the heat equation (2.1) in the shearlet domain, indeed.

By solving (2.15) for each j, k, m, we can obtain $\widehat{U}$ by substituting $C_{j,k,m}$ in (2.5). Finally by applying n-D inverse Fourier transform to $\widehat{U}$ we can get the solution.

3. Example - Formulation for two-dimensional heat equation

In this section, as an example of the developed procedure, we obtain the shearlet formulation for the solution of two-dimensional heat equation. First, we consider two-dimensional heat equation as

$$u_t = {c'}^2(u_{xx} + u_{yy}), \ 0 \le x < a, \ 0 \le y < b \tag{3.1}$$

$$\begin{cases} u(0,y,t) = u(a,y,t) = 0, \ 0 \le y \le b, \ t \ge 0 \\ u(x,0,t) = u(x,b,t) = 0, \ 0 \le x \le a, \ t \ge 0 \end{cases} \tag{3.2}$$

For solving the above problem with shearlet system method, firstly, we apply 2-D Fourier transform with respect to variables x,y to (3.1) and get:

$$\frac{\partial}{\partial t}\widehat{U}(\xi_1,\xi_2) = -c'^2(\xi_1^2\widehat{U} + \xi_2^2\widehat{U}). \tag{3.3}$$

Then we consider

$$u(x,y,t) = \sum_{j,k,m} C_{j,k,m}\tilde{\psi}_{j,k,m}(x,y,t),$$

and

$$\Delta u(x,y,t) = \sum_{j,k,m} C^{\Delta}_{j,k,m}\tilde{\psi}_{j,k,m}(x,y,t),$$

where $C_{j,k,m} = \langle u(x,y,t), \tilde{\psi}_{j,k,m}(x,y,t)\rangle$ and $C^{\Delta}_{j,k,m} = \langle \Delta u(x,y,t), \tilde{\psi}_{j,k,m}(x,y,t)\rangle$.

Applying 2-D Fourier transform and using Plancherel theorem to (3.3) yields

$$\widehat{U}(\xi_1,\xi_2,t) = \sum_{j,k,m} C_{j,k,m}\hat{\tilde{\psi}}_{j,k,m}(\xi_1,\xi_2,t) \tag{3.4}$$

$$\widehat{\Delta u}(\xi_1,\xi_2,t) = \sum_{j,k,m} C^{\Delta}_{j,k,m}\hat{\tilde{\psi}}_{j,k,m}(\xi_1,\xi_2,t). \tag{3.5}$$

By replacing $\widehat{U}$ and $\widehat{\Delta u}$ into (3.3), we obtain

$$\pm i|\xi|c\sum_{j,k,m} C_{j,k,m}\hat{\tilde{\psi}}_{j,k,m}(\xi_1,\xi_2,t) = c'^2\sum_{j,k,m} C^{\Delta}_{j,k,m}\hat{\tilde{\psi}}_{j,k,m}(\xi_1,\xi_2,t). \tag{3.6}$$

Now, (3.6) can be rewritten as follows

$$\sum_{j,k,m}\left[\pm i|\xi|cC_{j,k,m} - c'^2C^{\Delta}_{j,k,m}\right]\hat{\tilde{\psi}}_{j,k,m} = 0, \tag{3.7}$$

with respect to the definition of $C_{j,k,m}, C^{\Delta}_{j,k,m}$ and some properties of inner product, we have

$$\sum_{j,k,m}\left[\langle \pm i|\xi|cu - c'^2\Delta u, \hat{\tilde{\psi}}_{j,k,m}\rangle\right]\hat{\tilde{\psi}}_{j,k,m} = 0. \tag{3.8}$$

since $\{\psi_{j,k,m}\}_{j,k,m}$ is a parseval frame, it could be concluded

$$\langle \pm i|\xi|cu - c'^2\Delta u, \hat{\tilde{\psi}}_{j,k,m}\rangle = 0,$$

therefore

$$\pm i|\xi|cC_{j,k,m} - c'^2C^{\Delta}_{j,k,m} = 0. \tag{3.9}$$

Now, by using the definition of $C_{j,k,m}, C^{\Delta}_{j,k,m}$ and the Plancherel theorem, we get

$$\begin{aligned} C_{j,k,m} &= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{u}(\xi).\overline{\hat{\psi}}_{j,k}(\xi)e^{i\langle m,\xi\rangle}d\xi,\ \xi=(\xi_1,\xi_2), \\ C^{\Delta}_{j,k,m} &= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{\Delta u}.\overline{\hat{\psi}}_{j,k}(\xi)e^{i\langle m,\xi\rangle}d\xi. \end{aligned} \tag{3.10}$$

By using the Fourier transform property for the drivatives of function u, we have

$$\widehat{\Delta U} = -\xi_1^2\widehat{U} - \xi_2^2\widehat{U} = -(\xi_1^2+\xi_2^2)\widehat{U}. \tag{3.11}$$

Hence $C^{\Delta}_{j,k,m}$ can be rewritten as follows:

$$C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}(\xi_1^2+\xi_2^2)\hat{u}(\xi)\cdot\overline{\hat{\psi}}_{j,k}(\xi)e^{i\langle m,\xi\rangle}d\xi.$$

Now, by substituting $S^T_{-k}A_{2^{-j}}\xi$ with ξ, we have:

$$C_{j,k,m} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A_{2^j}S^T_k\left(\hat{u}(\xi).\overline{\hat{\psi}}_{j,k}(\xi)\right)e^{i\langle m,\xi\rangle}d\xi,$$

$$C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} |A_{2^j}S^T_k\xi|^2 A_{2^j}S^T_k\left(\hat{u}(\xi).\overline{\hat{\psi}}_{j,k}(\xi)\right)e^{i\langle m,\xi\rangle}d\xi.$$

For simplicity, we consider $\Gamma := A_{2^j}S^T_k\left(\hat{u}(\xi).\overline{\hat{\psi}}_{j,k}(\xi)\right)$, hence, $C_{j,k,m}$ and $C^{\Delta}_{j,k,m}$ can be rewritten as follows

$$C_{j,k,m} = \frac{1}{(2\pi)^2}\int \Gamma e^{i\langle m,\xi\rangle}d\xi, \tag{3.12}$$

and

$$\begin{aligned} C^{\Delta}_{j,k,m} &= -\tfrac{1}{(2\pi)^2}\int\left(2^{2j}\xi_1^2 + 2^j(k\xi_1+\xi_2)^2\right)\Gamma d\xi \\ &= -\tfrac{1}{(2\pi)^2}\int\left(2^{2j}\xi_1^2 + 2^jk^2\xi_1^2 + 2^{j+1}k\xi_1\xi_2 + 2^j\xi_2^2\right)\Gamma d\xi \\ &= -\tfrac{1}{(2\pi)^2}\int 2^j\left(2^j\xi_1^2 + k^2\xi_1^2 + 2k\xi_1\xi_2 + \xi_2^2\right)\Gamma d\xi \\ &= -\tfrac{2^j}{(2\pi)^2}\int\left((2^j+k^2)\xi_1^2 + 2k\xi_1\xi_2 + \xi_2^2\right)\Gamma d\xi. \end{aligned} \tag{3.13}$$

It could be seen that $C^{\Delta}_{j,k,m}$ is a combination of the following terms

$$C^{\Delta}_{k_1} = 2^j(2^j + k^2)(\tfrac{\partial^2 C_{j,k,m}}{\partial m_1^2}),$$

$$C^{\Delta}_{k_2} = 2^j(\tfrac{\partial^2 C_{j,k,m}}{\partial m_2^2}), \tag{3.14}$$

$$C^{\Delta}_{k_1,k_2} = 2^j(2k)(\tfrac{\partial^2 C_{j,k,m}}{\partial m_1 \partial m_2}).$$

Replacing (3.14) and (3.12) in (3.9) leads to

$$i|\xi|cC_{j,k,m} = c'^2 2^j \left[(2^j + k^2)(\frac{\partial^2 C_{j,k,m}}{\partial m_1^2}) + (\frac{\partial^2 C_{j,k,m}}{\partial m_2^2}) + (2k)(\frac{\partial^2 C_{j,k,m}}{\partial m_1 \partial m_2})\right]. \tag{3.15}$$

Indeed, (3.15) is heat equation (3.1) in the shearlet domain.

Let $A := 2^j(2^j + k^2),\ B := k2^j,\ D := 2^j$, then we have

$$B^2 - AD = k^2 2^{2j} - 2^j(2^j + k^2)2^{2j} = -2^{3j} < 0,$$

which states that (3.15) is an elliptic partial differential equation. Now, by solving the following time independent equation for each j, k, m,

$$i|\xi|cC_{j,k,m} = c'^2 \left[A(\frac{\partial^2 C_{j,k,m}}{\partial m_1^2}) + D(\frac{\partial^2 C_{j,k,m}}{\partial m_2^2}) + 2B(\frac{\partial^2 C_{j,k,m}}{\partial m_1 \partial m_2})\right]. \tag{3.16}$$

We can obtain $\widehat{U}$ by substituting $C_{j,k,m}$ in (3.4). Finally, by applying 2-D inverse Fourier transform to $\widehat{U}$, we can get the solution.

4. Conclusion

A method for solution of n-dimensional transient heat equation by making use of shearlet frames is presented and to better clarify the method an example in two dimensions is presented. This approach is general and can be employed for other PDE problems such as Poisson and wave equation. As it was shown, in this approach the unknown function is approximated by an expansion via shearlet frame expansion and the coefficients of this expansion are obtained by employing Fourier transformation and Planchere theorem. The main merit of this approach is that for finding the unknown coefficients there is no need to solve a system of simultaneous algebric equations and each of the expansion coefficients can be obtained from a separte time independent differential equation. This property is of extreme applicability importance as the user of the method can step by step increase the accuracy of the solution to whatever degree that is satisfactory.

References

1. Amin Khah, M., Askari Hemmat, A. and Raisi Tousi, R. Integral representation for solutions of the wave equation by shearlets, Optik-International Journal for Light and Electron Optics, 127(22), 10554–10560 (2016).
2. G. Chen, Semi-analytical solutions for 2-D modeling of long pulsed laser heating metals with temperature dependent surface absorption, Optik, International Journal for Light and Electron Optics, 2017.
3. S. Dahlke, S. Hauser, G. Steidl and G. Teschke, Shearlet coorbit spaces: traces and embeddings in higher dimensions. Monatshefte fur Mathematik, 169(1), 15-32 (2013).
4. W. Dai and R. Nassar, A compact finite difference scheme for solving a three-dimensional heat transport equation in a thin film, Numer. Methods Partial Differ. Equ., 16, 441–458 (2000).
5. B. Fornberg A Practical Guide to Pseudospectral Methods. Cambridge University Press, Cambridge, UK (1996).
6. K. Guo, G. Kutyniok, D. Labate, Sparse multidimensional representations using anisotropic dilation and shear operators, Wavelets and Splines, Nashboro Press, Nashville, 189201 (2006).
7. J. Hesthaven, S. Gottlieb and D. Gottlieb, "Spectral methods for time-dependent problems", Cambridge UP, Cambridge, UK (2007) .
8. Kaiser, G. A Friendly Guide to Wavelets, Boston: Birkhauser, (1994).
9. Z. Kalateh Bojdi and A. Askari Hemmat, Wavelet collocation methods for solving the Pennes bioheat transfer equation, Optik, International Journal for Light and Electron Optics, 132, 80–88 (2017).
10. Z. Kargar and H. Saeedi, B-spline wavelet operational method for numerical solution of time-space fractional partial differential equations, Int. J. Wavelets Multiresolut. Inf. Process., 15, 1750034 (2017).
11. A. Kaw and E. Eric Kalu, Numerical Methods with Applications, Contains a brief, engineering-oriented introduction to FDM (for ODEs) in Chapter 08.07 (2008).
12. G. Kutyniok, D. Labate, Shearlets: Multiscale analysis for multivariate data, Springer Science & Business Media, 330 (2012).
13. G. Kutyniok, J. Lemvig and W. Q. Lim, Optimally sparse approximations of 3D functions by compactly supported shearlet frames, SIAM Journal on Mathematical Analysis, 44(4), 29623017 (2012).
14. K.W. Morton and D.F. Mayers, Numerical Solution of Partial Differential Equations, An Introduction. Cambridge University Press, (2005).
15. J. N. Reddy, An Introduction to the Finite Element Method (Third ed.). McGraw-Hill. ISBN 9780071267618 (2006).
16. G.D. Smith, Numerical Solution of Partial Differential Equations Finite Difference Methods, Third ed., Oxford, Oxford University Press, (1985).
17. J. Strikwerda, Finite Difference Schemes and Partial Differential Equations (2nd ed.). SIAM. ISBN 978-0-89871-639-9 (2004).
18. B. Sun, et al., Solving wave equations in the curvelet domain: A multi-scale and multi-directional approach, Journal of Seismic Exploration, 18(4), 385–399 (2009).

19. WH. Teukolsky, SA. Vetterling, WT. Flannery, "Section 20.7. Spectral Methods". Numerical Recipes: The Art of Scientific Computing (3rd ed.). New York: Cambridge University Press. ISBN 978-0-521-88068-8 BP (2007).
20. O. C. Zienkiewicz, Robert L Taylor, J.Z. Zhu The Finite Element Method: Its Basis and Fundamentals. Butterworth-Heinemann. ISBN 978-0-08-095135-5 (31 August 2013).