ON THE PERTURBATION OF FRAMES

N.RAMEZANZADE *, 1 A.A. AREFIJAMAAL 2

1 Department of Mathematics,Hakim Sabzevari University, Sabzevar, Iran
Nramezanzade@yahoo.com
2 Department of Mathematics,Hakim Sabzevari University, Sabzevar, Iran
Areffijamaal@hsu.ac.ir

ABSTRACT. There are several results on the perturbation of frame sequences in Hilbert spaces that have introduced by Christensen. We state some results on the perturbation of dual frames.

1. Introduction

A sequence \{f_i\} in a Hilbert space H is called a frame if there exist constants A, B > 0 such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad (f \in H).$$

The numbers A and B are called frame bound, also they are not unique. In addition, it follows from the definition that if \{f_i\} is a frame for H, then \(\overline{\text{span}}\ \{f_i\} = H\). The frame is called tight frame when A = B. If A = B = 1, it is called a Parseval frame. We say that \{f_i\} is a frame sequence if it is a frame for \(\overline{\text{span}}\ \{f_i\}\). The sequence \{f_i\} \subseteq H is a Bessel sequence if at least the upper frame bound B exists. In this case the bounded operator \(T : l^2(\mathbb{N}) \rightarrow H\) defined by \(T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i\) is usually called the pre-frame operator. Furthermore, the composing

2010 Mathematics Subject Classification. Primary: 47H10; Secondary: 47H09.
Key words and phrases. Frames; Dual frames; Perturbations; Riesz bases.

* Speaker.
of operator $T$ with its adjoint gives the frame operator

$$S : H \to H, \quad Sf := TT^*f = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle f_i.$$ 

If both of the frame conditions are satisfied, then $S$ is invertible and self-adjoint. Moreover, the following holds

$$f = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i \rangle S^{-1}f_i.$$ 

A sequence $\{g_i\}_{i=1}^{\infty} \subseteq H$ is called a dual frame for $\{f_i\}_{i=1}^{\infty}$ if

$$f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i, \quad (f \in H).$$

The classical choice for $\{g_i\}_{i=1}^{\infty}$ is $\{S^{-1}f_i\}_{i=1}^{\infty}$. Every frame at least has a dual. In fact, if $\{f_i\}_{i=1}^{\infty}$ is a frame, then $\{S^{-1}f_i\}_{i=1}^{\infty}$, which is a frame with bounds $B^{-1}$ and $A^{-1}$, is a dual for $\{f_i\}_{i=1}^{\infty}$; it is called the canonical dual, a dual which is not be the canonical dual is called an alternate dual, or simply a dual.

**Example 1.1.** [2] Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $H$ and consider the frame

$$\{f_i\}_{i=1}^{\infty} = \{e_1, e_2, e_3, \cdots \},$$

which is a frame with bounds $A = 1, B = 2$. The canonical dual frame is given by

$$\{S^{-1}f_i\}_{i=1}^{\infty} = \left\{ \frac{1}{2} e_1, \frac{1}{2} e_1, e_2, e_3, \cdots \right\}.$$ 

As an example of non-canonical dual frame we mention

$$\{g_i\}_{i=1}^{\infty} = \{0, e_1, e_2, e_3, \cdots \}$$

and

$$\{g_i\}_{i=1}^{\infty} = \left\{ \frac{1}{3} e_1, \frac{2}{3} e_1, e_2, e_3, \cdots \right\}.$$ 

**Theorem 1.2.** [1] Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $H$ with the pre-frame operator $T$. Then $\{g_i\}_{i=1}^{\infty}$ is a dual for $\{f_i\}_{i=1}^{\infty}$ if and only if

$$g_i = S^{-1}f_i + u_i$$

for some Bessel sequence $\{u_i\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} \langle f, f_i \rangle u_i = 0, \quad (f \in H).$$
2. Main results

The question of stability where plays a crucial role states that if \( \{e_i\} \) is an orthonormal basis and \( \{g_i\} \) in some sense is "close" to \( \{e_i\} \), does it follow that \( \{g_i\} \) also is a basis? A classical result states that if \( \{e_i\} \) is an orthonormal basis for a \( H \), then a sequence \( \{g_i\} \) in \( H \) is an orthonormal basis if there exists a constant \( \lambda \in (0, 1) \) such that

\[
\| \sum \limits_{i=1}^{n} c_i(e_i - g_i) \| \leq \| \sum \limits_{i=1}^{n} c_i e_i \|,
\]

for all finite sequences of scalars \( \{c_i\} \). We will now discuss a natural extension of this result to the frame setting. That is, assuming that \( \{f_i\} \) is a frame for a Hilbert space \( H \), we want to find conditions on a perturbed family \( \{g_i\} \) that imply that it is a frame. We discuss also similar results for dual frames.

We apply the following lemma frequently.

**Lemma 2.1.** [4] Let \( X \) be a Banach space and \( U : X \rightarrow X \) a linear operator, if constants \( \lambda_1, \lambda_2 \in (0, 1) \) there exist such that

\[
\| Ux - x \| \leq \lambda_1 \| x \| + \lambda_2 \| Ux \|, \quad (x \in X),
\]

then the linear operator \( U \) is bounded and invertible. In addition,

\[
\frac{1 - \lambda_1}{1 + \lambda_2} \| x \| \leq \| Ux \| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \| x \|, \quad (x \in X).
\]

and

\[
\frac{1 - \lambda_2}{1 + \lambda_1} \| x \| \leq \| U^{-1}x \| \leq \frac{1 + \lambda_2}{1 - \lambda_1} \| x \|, \quad (x \in X).
\]

**Theorem 2.2.** [3, 5, 6] Let \( \{f_i\}_{i=1}^{\infty} \) be a frame with bounds \( A, B \) for \( H \). Let \( \{g_i\}_{i=1}^{\infty} \subseteq H \) and suppose that there exist constants \( \lambda_1, \lambda_2, \mu \geq 0 \) such that \( \max (\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2) < 1 \) and

\[
\| \sum \limits_{i=1}^{n} c_i(f_i - g_i) \| \leq \lambda_1 \| \sum \limits_{i=1}^{n} c_i f_i \| + \lambda_2 \| \sum \limits_{i=1}^{n} c_i g_i \| + \mu \left( \sum \limits_{i=1}^{n} | c_i |^2 \right)^{\frac{1}{2}}
\]

for all scalars \( c_1, c_2, \ldots, c_n (n \in \mathbb{N}) \), then \( \{g_i\}_{i=1}^{\infty} \) is a frame with bounds

\[
A \left( 1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1 + \lambda_2} \right)^2 B \left( 1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1 - \lambda_2} \right)^2.
\]

**Theorem 2.3.** [3, 5, 6] Suppose that \( \{g_i\}_{i=1}^{\infty} \subseteq H \) and a sequence \( \{f_i\}_{i=1}^{\infty} \) be a frame for \( \text{span}\{f_i\}_{i=1}^{\infty} \), with bounds \( A, B \). If there exists
constant $\lambda_1, \mu \geq 0$, $\lambda_2 \in [0, 1]$ and scalars $c_1, c_2, \ldots, c_n \in \mathbb{N}$ such that,
\[
\| \sum_{i=1}^{n} c_i (f_i - g_i) \| \leq \lambda_1 \| \sum_{i=1}^{n} c_i f_i \| + \lambda_2 \| \sum_{i=1}^{n} c_i g_i \| + \mu \left( \sum_{i=1}^{n} | c_i |^2 \right)^{\frac{1}{2}},
\]
then \( \{g_i\}_{i=1}^{\infty} \) is a Bessel sequence with bound $B \left( 1 + \frac{\lambda_1 + \lambda_2 + \sqrt{\mu}}{1 - \lambda_2} \right)^2$.

**Theorem 2.4.** [3, 5, 6] Assume that \( \{g_i\}_{i=1}^{\infty} \subseteq H \) and a sequence \( \{f_i\}_{i=1}^{\infty} \) is a frame for $H$ with bounds $A, B$ and $K$ is a compact operator from $l^2$ into $H$ with
\[
Kc_i = \sum_{i=1}^{\infty} c_i(f_i - g_i).
\]
Then the sequence \( \{g_i\}_{i=1}^{\infty} \) is a frame for $\text{span} \{g_i\}_{i=1}^{\infty}$.

**Theorem 2.5.** Suppose that \( \{f_i\}_{i=1}^{\infty} \) is a frame, then there exists an infinite dual \( \{g_i\}_{i=1}^{\infty} \subseteq H \) such that
\[
Kc_i = \sum_{i=1}^{\infty} c_i(g_i - S_{F}^{-1} f_i),
\]
is a compact operator from $l^2$ to $H$.

**Theorem 2.6.** Let \( \{g_i\}_{i=1}^{\infty} \) be a dual frame of \( \{f_i\}_{i=1}^{\infty} \) and \( \{h_i\}_{i=1}^{\infty} \) be a Bessel sequence such that
\[
Kc_i = \sum_{i=1}^{\infty} c_i(g_i - h_i).
\]
Then \( \{h_i\}_{i=1}^{\infty} \) is also a dual of \( \{f_i\}_{i=1}^{\infty} \).

**REFERENCES**